Disk Entanglement Entropy for an Abelian Gauge Field

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Based on:
(Agón,Headrick,Jafferis,Kasko-2013)
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CONTENT

MOTIVATION

Calculations of EE has been of special interest in different contexts

- Encodes fundamental information of the underline quantum field theory (connected to c,f and a-theorems)
- Characterization and discovery of quantum phase transitions
  - Fundamental quantity in Information Theory (quantum computers?)
- In the holographic context It seems to be a key quantity of study to understand quantum gravity

MOTIVATION

Summary

In this work we outline the calculation of the vacuum disk entanglement entropy for a pure Maxwell Field with a compact $U(1)$ gauge group.

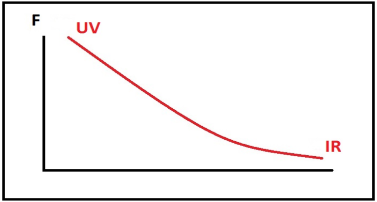
- Maxwell Field $\leftrightarrow$ Compact Free Scalar (is trivial enough to allow this calculation)
- In 3 dimension $F(r_D) := r_D S'(r_D) - S(r_D)$ (its proof is not valid in general [Casini,Huerta-2012]) (gauge theories?)
- In the context of the replica trick $\lim_{n\to 1} S_n \to S$, how to implement analytic/numerical continuation?

MOTIVATION

F Theorem

$$F(r_D) := r_D S'(r_D) - S(r_D)$$

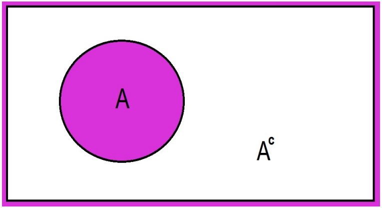
i It is ultraviolet finite and scheme independent
ii For CFT's it is constant and $= (F_{S^3} = -\ln Z_{S^3})$
iii It is monotonically decreasing (F-theorem)



- Codifies Topological and Local dof (i.e for Chern-Simons theory $F = \frac{1}{2} \ln k$)
- Discovery of exotic flows $N_I^{IR} > N_I^{UV}$ as long as $N_I^{IR} < F^{UV}$? (i.e C-S + Maxwell in the UV$\sim$ Maxwell, IR$\sim$ CS $\rightarrow$ $F^{UV} > \frac{1}{2} \ln k$ for every k)
- Provide extra evidence of the F theorem by computing $F(r_D)$

Entanglement Entropy

Time slice of the 3d space-time



- $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{A^c}$
- The Entanglement entropy between A and A^c

$$S(A) = -\operatorname{Tr} \rho_A \ln \rho_A \,,$$

- where $\rho_A = \mathrm{Tr}_{A^c}\rho$ and $\rho = |0\rangle\langle 0|$

Path integral representations

As usual (wick rotation) $\beta = \tau = it$, $\tau =$ Euclidean time and $(\beta = 1/T)$, $S_E = \int_0^\beta L_E d\tau$.

$$|0\rangle\langle 0| \approx \lim_{T\to 0} \rho = \lim_{T\to 0} \frac{e^{-\beta H}}{\mathrm{Tr}\, e^{-\beta H}}$$

Matrix elements of ρ

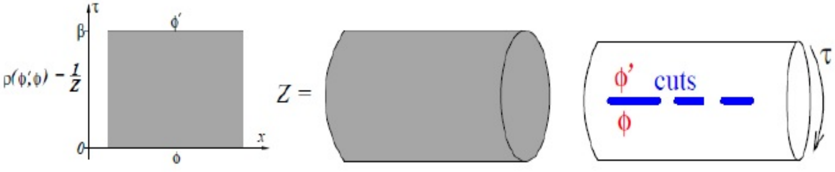
$$\rho(\phi', \phi'') = \langle \phi' | \rho | \phi'' \rangle = \frac{1}{\mathrm{Tr}\, e^{-\beta H}} \langle \phi' | e^{-\beta H} | \phi'' \rangle,$$

$$\text{where} \qquad Z(\beta) = \int d\phi \langle \phi | e^{-\beta H} | \phi \rangle \, .$$

$$\rho(\phi', \phi'') = Z(\beta)^{-1} \int [d\phi(y,\tau)] \delta(\phi(y,0) - \phi') \delta(\phi(y,\beta) - \phi'') e^{-S_E} ,$$

The Reduced density matrix

$$\begin{aligned} \rho_A(\phi', \phi'') &= Z(\beta)^{-1} \int [d\phi(y,\tau)] \delta(\phi(y\epsilon A, 0) - \phi') \delta(\phi(y\epsilon A, \beta) - \phi'') \\ &\qquad \delta(\phi(y\epsilon A^c, 0) - \phi(y\epsilon A^c, \beta)) e^{-S_E} . \end{aligned}$$



Rényi Entropies and Replica Trick

All quantum entanglement information ρ_A is encoded in

$$S_n(A) = \frac{1}{1-n} \ln \mathrm{Tr}\, \rho_A^n,$$

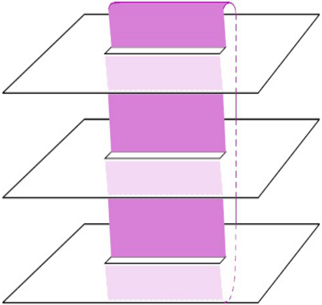
with $S(A) = \lim_{n\to 1} S_n(A)$. We need

$$\mathrm{Tr}\, \rho_A^n$$

For instance

$$\rho_A^3(\phi, \phi') = \int [d\phi''][d\phi''']$$
$$\times \rho_A(\phi, \phi'')\rho_A(\phi'', \phi''')\rho_A(\phi''', \phi')$$

n sheeted branch covering of $\mathbf{R}^d$



Replica Trick

Made n replicas of the fields

Related by

$$\hat{\phi}_j(x, \tau = \beta_-) = \hat{\phi}_{j+1}(x, \tau = 0_+)$$

with $n + 1 = 1$, for x on A

We can interpret it as a single ϕ living in an n sheeted Riemann surface $\mathcal{R}_n$.

For free theories, move to its Fourier space

$$\tilde{\phi}^k(x) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} e^{2\pi ijk/n} \hat{\phi}_j(x) \,, \qquad (k = 1, \ldots, n) \,, \quad \text{and} \quad x \,\epsilon\, \mathbf{R}^d$$

for which the monodromy conditions read simply

$$\tilde{\phi}^k(x, \tau = \beta_-) = e^{-2\pi ik/n} \tilde{\phi}^k(x, \tau = 0_+) \,.$$

In those cases

$$I[\phi] = \int_{\mathcal{R}_n} d^{d+1}x \mathcal{L}[\phi] = \sum_{k=1}^{n} \int_{\mathbf{R}^d} d^{d+1}x \tilde{\mathcal{L}}[\tilde{\phi}^k]. \tag{1}$$

If we denote the partition function on this geometry as $Z_n(A)$, then $\mathrm{Tr}\,\rho_A^n = Z_n(A)/Z^n$ and the Rény entropies are given by

$$S_n(A) = \frac{\ln Z_n(A) - n \ln Z}{1-n}, \tag{2}$$

Analytically continue to $n = 1$ which leads to $S(A)$

Duality

The Maxwell theory in 3 dimensions turns out to have a dual description in terms of a compact free scalar

$$F_{ij} = \epsilon_{ijk}\partial^k\phi.$$

The Maxwell equations for F and the Laplace equation ϕ are related as

$$\partial^i F_{ij} = 0 \qquad \rightarrow \qquad \epsilon_{ijk}\partial^i\partial^k\phi = 0\,,$$

$$\partial_i F^{*i} = 0 \qquad \rightarrow \qquad \nabla^2\phi = 0\,,$$

where

$$F^{*i} = \epsilon^{ijk}F_{jk}$$

Even though classically they match, the equivalence at a quantum level is more subtle and just hold for both compact theories $2\pi R = g$, where

$$\phi \sim \phi + 2\pi Rn \quad \text{and} \quad \psi \sim \psi + m/R$$

where we have written the gauge transformation as

$$A_\mu \sim A_\mu + \partial_\mu \psi$$

and therefore the non-compact limits are complementary
$R \to \infty$ non-compact free scalar theory (IR)
$1/R \to \infty$ non-compact free Maxwell theory (UV)

- The F function should interpolate between those cases. The dimensionless relevant parameter is $(2\pi R)^2 r_D = g^2 r_D$
- Check the F theorem $F(r_D)$
  - The IR of F is $F = (\ln 2)/8 - 3\zeta(3)/(16\pi^2)$ [Klebanov,Pufu,Safdi-2011]
  - In [Klebanov,Pufu,Sachdev,Safdi-2012], $F_{S^3} = \frac{\zeta(3)}{4\pi^2} - \frac{1}{2}\ln(r_S g^2)$ ($F_{S^3} = -\ln Z_{S^3}$) similar to the UV of F.

Renyi entropies for a Maxwell Field

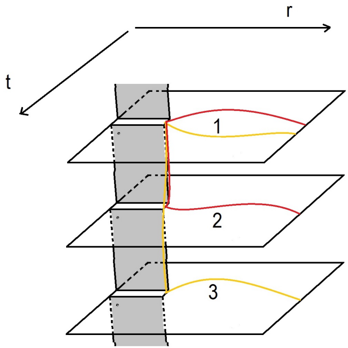
To perform our computation we will follow the replica trick

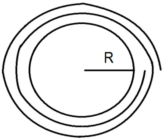
$$I[\phi] = \frac{1}{2}\int d^3x \partial_\mu \phi \partial^\mu \phi$$

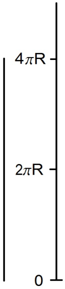
- Due to SSB the fields get a definite value at infinity
- The periodicity implies $0 = 2\pi nR$ for ϕ
- The computation of the path integral involves a sum over winding sectors (Instanton sum)
- Since the theory is free we can diagonalize the monodromy
- Find the classical solutions for the action is equivalent to solving for the capacity of an electrostatic system

Renyi entropies for a Maxwell Field

Asymptotic constant value at ∞, But since $0 = 2\pi n R$ we have different non-equivalent classical solutions







We can visualize the different classical solutions by looking at the topology of the Riemann surface

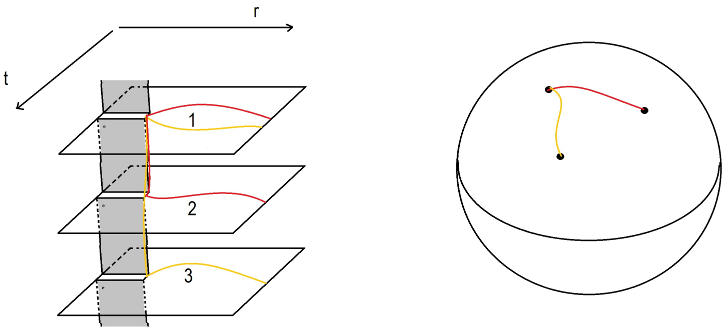


Figure: 3 sheeted surface representation of $\mathrm{Tr}\,\rho_{r_D}^3$ and its topological equivalent surface

For n-sheets, then we will have $n-1$ topological inequivalent trajectories connecting the different infinities

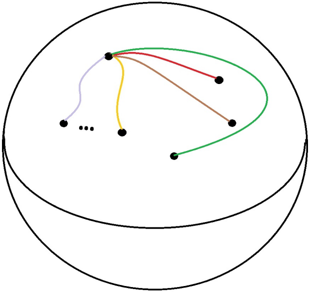
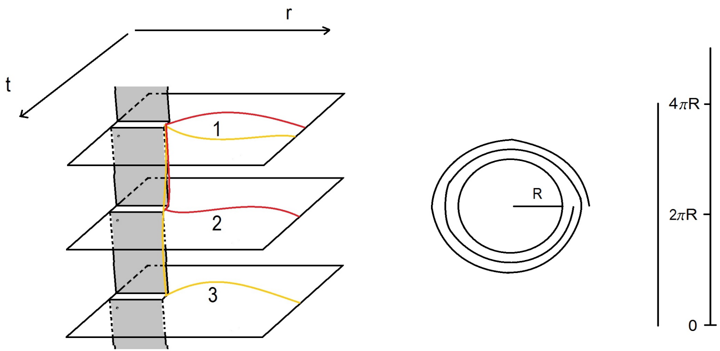


Figure: 3 sphere without n-points

For computational purposes (to evaluate the path integral) is easier to consider that the scalar is not compact and evaluate the action for all the possible asymptotic boundary conditions



- Choosing

$$\phi_n \to 0$$

  we have that for the $(n-1)$ remaining sheets the field could be

$$\phi_j \to 2\pi R w_j .$$

- Different classical sols will be ϕ_w, with $w = (w^1, \ldots, w^{n-1})$
- Therefore any field conf can be separated uniquely $\phi = \phi_{cl} + \phi_{qu}$
- Since $I[\phi] = \frac{1}{2}\int d^3x \partial_\mu \phi \partial^\mu \phi$ then $I[\phi_{cl}] + I[\phi_{qu}]$

$$Z_n(r_D) = \int [d\phi] e^{-I[\phi]} = Z_n^{\text{cl}}(r_D) Z_n^{\text{qu}}(r_D) , \tag{3}$$

implies

$$S_n(r_D) = S_n^{\text{qu}}(r_D) + S_n^{\text{cl}}(r_D) . \tag{4}$$

The classical partition function involves the sum over all winding numbers

$$Z_n^{\mathrm{cl}}(r_D) = \sum_{w \in \mathbf{Z}^{n-1}} e^{-I[\phi_w]} \,, \tag{5}$$

Since $I[\phi_w] \propto (\partial_i \phi_w)^2$ and $\phi_w \sim \phi_w + 2\pi R n_w$
we can redefined a field $\phi_w = 2\pi R \hat{\phi}_w$ and factorize $(2\pi R)^2$ from $I[\phi_w]$, or $r \equiv (2\pi R)^2 r_D$ and put the disk at $r_D = 1$

- Going to the Fourier space

$$\tilde{\phi}_w^k \to \tilde{w}^k = \frac{1}{\sqrt{n}} \sum_{j=1}^{n-1} e^{2\pi ijk/n} w^j$$

since $\phi_j \to w_j$

- Factorizing the asymptotic value of the fields, the action will be

$$I[\hat{\phi}_w] = \sum_{k=1}^{n-1} |\tilde{w}^k|^2 J(\frac{k}{n}) = \frac{1}{n} \sum_{j,j',k=1}^{n-1} w^j w^{j'} e^{2\pi i(j-j')k/n} J(\frac{k}{n}),$$

where $J(k/n) = \frac{1}{2} \int d^3x \partial^\mu \tilde{\phi}_{k/n} \partial_\mu \tilde{\phi}_{k/n}$ with $\tilde{\phi}_{k/n} \to 1$

To find $I[\hat{\phi}_w]$ we need to find the action $J(k/n)$ of a free scalar ϕ with

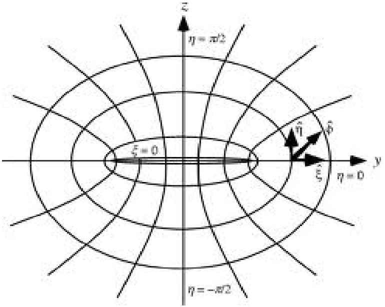
$$\tilde{\phi}_{k/n} \to 1$$

and monodromy

$$\tilde{\phi}_{k/n}(x \,\epsilon\, D, \tau = 0_-) = e^{-2\pi i k/n}\tilde{\phi}_{k/n}(x \,\epsilon\, D, \tau = 0_+)\,.$$

$$\partial_\tau\tilde{\phi}_{k/n}(x \,\epsilon\, D, \tau = 0_-) = e^{-2\pi i k/n}\partial_\tau\tilde{\phi}_{k/n}(x \,\epsilon\, D, \tau = 0_+)\,.$$

A small visualization for the boundary conditions



- We can write formal expressions for the classical Rényis

$$Z_n^{\mathrm{cl}}(r) = \Theta(irM_n/\pi)\,, \qquad S_n^{\mathrm{cl}}(r) = -\frac{1}{n-1}\ln\Theta(irM_n/\pi)\,, \tag{6}$$

where Θ is defined as

$$\Theta(A) = \Theta(0|A) = \sum_{w\in\mathbf{Z}^{n-1}} e^{\pi i A_{jj'} w^j w^{j'}}\,, \tag{7}$$

and M_n is given by

$$(M_n)_{jj'} = \frac{1}{n}\sum_{k=1}^{n-1} e^{2\pi i(j-j')k/n} J(\frac{k}{n})\,. \tag{8}$$

Limiting values of S(r)

After some work we found that

$$J(k/n) = 2\pi(1 - 2k/n)\tan(\pi k/n)$$

and with that $S_n(r)$ for $n \geq 1$. However it is not clear how to obtain $S_1(r)$.

- For large r, $S_n(r) \approx S_n^{\mathrm{nc}}(r)$, $S(r) \approx S^{\mathrm{nc}}(r)$ and

$$F(r) \approx F^{\mathrm{nc}} = \frac{\ln 2}{8} - \frac{3\zeta(3)}{16\pi^2}$$

- For small r,

$$Z_n^{\mathrm{cl}}(r) = \sum_w \exp\left(-r(M_n)_{jj'} w^j w^{j'}\right) \approx \int d^{n-1}w \cdots$$
$$\approx (r/\pi)^{(1-n)/2} (\det M_n)^{-1/2} .$$

and

$$\Delta S_n(r) \approx \frac{1}{2} \ln\left(\frac{4r}{n}\right) + \frac{1}{n-1} \ln\left(\frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}}\right) .$$

The limit $n \to 1$ can be taken

$$\Delta S(r) \approx \frac{1}{2} \ln r - \frac{\gamma}{2} ,$$

$$F(r) \approx \frac{\ln 2}{8} - \frac{3\zeta(3)}{16\pi^2} + \frac{1}{2} + \frac{\gamma}{2} - \frac{1}{2} \ln r .$$

Numerical extrapolation

- We would like to obtain $S_n(r)$ for $n = 1$. But that implies $\lim_{n\to 1} \Theta(irM_n/\pi)$ with (M_n a $(n-1)\times(n-1)$ matrix).
- We propose a rational interpolation of order (p, q), and tested with $S_2(r)$
- We also reproduce the constant piece s_1 of the UV limit of $S(r)$
- Writing $\Delta S_n(r) \approx \frac{1}{2}\ln r + s_n$, $\qquad s_n = \frac{1}{2}\ln\frac{4}{n} + \frac{1}{n-1}\ln\left(\frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}}\right)$

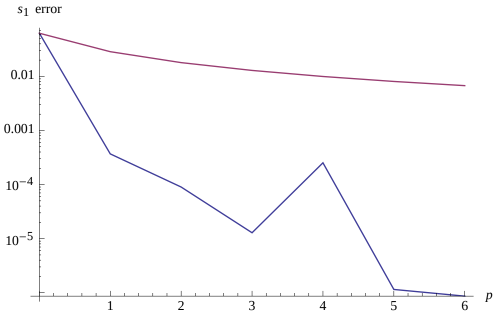


Figure: Absolute value of the error in the predicted value of s_1 from (top curve) degree $2p$ polynomial interpolating functions and (bottom curve) degree (p, p) rational interpolating functions, taking as input the values of s_n for $n = 2, \ldots, 2p + 2$.

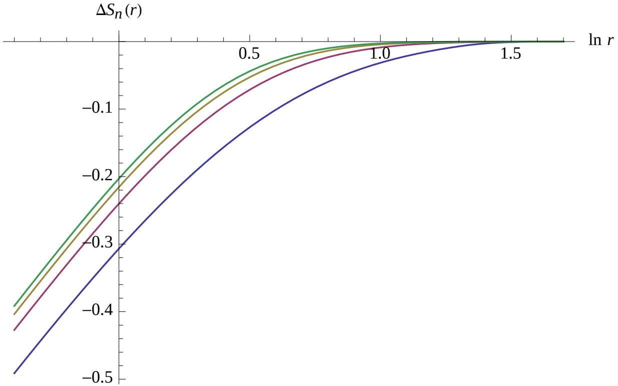


Figure: $\Delta S_{\mathrm{n}}(r) = S_{\mathrm{n}}^{\mathrm{cl}}(r)$ for $\mathrm{n} = 1, 2, 3, 4$ (bottom to top); $\Delta S(r) = S_1^{cl}(r)$ is obtained by the rational extrapolation method.

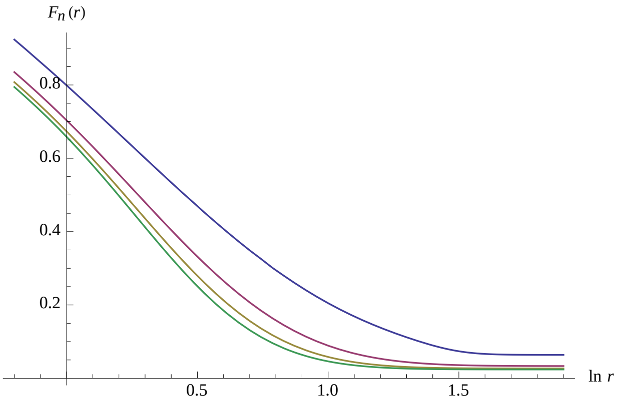


Figure: $\Delta F(r) = F^{cl}(r)$, calculated from $\Delta S(r) = S_1(r)$.

- It might be useful for other cases in particular the two disjoint interval case of a free compact boson [Calabrese, Cardy and Tonni-2009]

Determination of $J(\beta)$

$J(\beta)$ we want to compute the action of a single field that asymptotes to 1 at infinity and has the monodromy

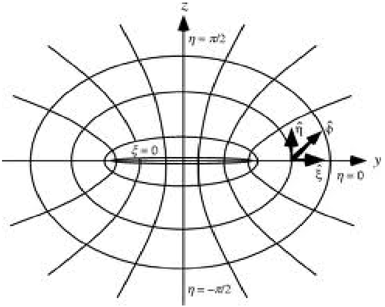
$$\begin{aligned} \phi(x \,\epsilon\, D, \tau = 0_-) &= e^{-2\pi i\beta}\phi(x \,\epsilon\, D, \tau = 0_+)\,. \\ \partial_\tau\phi(x \,\epsilon\, D, \tau = 0_-) &= e^{-2\pi i\beta}\partial_\tau\phi(x \,\epsilon\, D, \tau = 0_+)\,. \end{aligned} \tag{9}$$

The action turns out to be given by

$$S[\phi] = \frac{1}{2}\int_\partial d^2x\, h^{1/2} n^\mu \phi^* \partial_\mu \phi\,. \tag{10}$$

Here h is the determinant of the induced metric on the surface and n^μ is an outward-directed normal vector.

Due to the geometry of the problem, it turns out to be convenient to use oblate spheroidal coordinates (ζ, η, φ)



The general solution of the field

$$\phi(\zeta,\eta) = 1 - i\sum_{l=0}^{\infty} c_l Q_l(i\zeta)P_l(\eta)\,, \tag{11}$$

and the action is completely determined by the first coefficient of the previous solution, namely

$$J(\beta) = 2\pi c_0\,. \tag{12}$$

We need to find c_0

$$A c_{even} = b, \qquad c_{even} = A^{-1} b. \tag{13}$$

We can define a projector vector $(u)_l = \delta_{l0}$
and get $c_0 = u^T A^{-1} b$.
The precise solution $J(\beta)$ is

$$J(\beta) = 4 u^T \left(I + \lambda^2 T \right)^{-1} u \,. \tag{14}$$

where $\lambda = \cot \pi\beta$, and T is

$$T_{ll'} = \begin{cases} \dfrac{2(2l+1)\frac{l'}{2}!\,\frac{l-1}{2}!\,[\psi(\frac{1-l}{2}) + \psi(1+\frac{l}{2}) - \psi(\frac{1-l'}{2}) - \psi(1+\frac{l'}{2})]}{(-1)^{(l+l')/2}\pi^2 \frac{l}{2}!\,\frac{l'-1}{2}!\,(l'-l)(l+l'+1)} \\ \dfrac{\psi_1(\frac{1-l}{2}) - \psi_1(1+\frac{l}{2})}{\pi^2} \qquad (l' = l) \end{cases} \tag{15}$$

(with l, l' even).

- Exploring $\lambda \approx 0$ the first few coefficients ($k = 0, 1$ analytically and $k > 1$ numerically) agree with

$$u^T T^k u = \frac{1}{2k+1} \quad \text{leads to} \quad u^T (I + \lambda^2 T)^{-1} u = \frac{\arctan \lambda}{\lambda}, \tag{16}$$

and implies the answer

$$J(\beta) = 2\pi(1 - 2\beta) \tan(\pi\beta) \,. \tag{17}$$

- Around $\lambda \to \infty$ the coefficients diverge

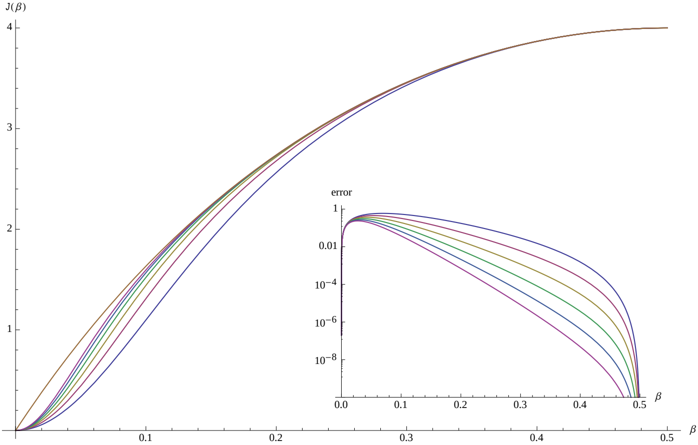


Figure: $J(\beta)$, analytic result (top curve) and from numerical inversion of truncated matrices for $\mathrm{d} = 2, 8, 32, 128, 512, 2048$.

Future Work

- Study other 3d flows i.e Maxwell + CS theory
- Generalize to compact scalar in higher dimensions which would be dual to higher dimensional gauge theories
- Apply this to Yang-Mills theory

THANKS