On time dependence and entanglement entropy

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`JHEP 1209(2012)055` E.C, Arnab Kundu
`JHEP 1401(2014)084` E.C, A.Kundu, J.Pedraza, W. Tangarife
`arXiv.1404.XXX`, E.C, A. Kundu, J. Pedraza, DL Yang

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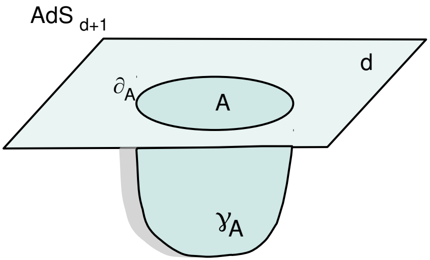
Outline

1. Motivation
   - Holographic Entanglement Entropy, Vaidya
     - Thermalization
     - Strong subadditivity and null energy condition
2. Collapse, weak field limit (in progress with A. Kundu, J. Pedraza and D.L. Yang)
3. Finite volume effects (in progress with J. Pedraza)
4. Conclusions and future directions

Holographic Entanglement entropy

Ryu, Tagayanagi 06

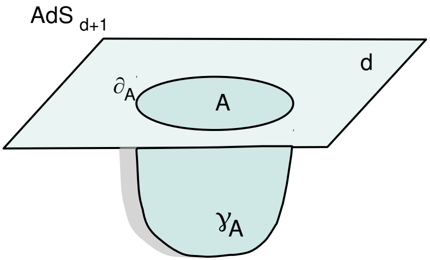


$$S_A = \frac{Area_{min}(\gamma_A)}{G_N^{d+1}}$$

Codimension 2 surface
Homology condition
$\gamma_A \sim A$
$\exists$ bulk region r s.t. $\delta r = \gamma_A \cup A$

Covariant prescription

Hubeny, Rangamani, Takayanagi 07



$$S_A = \frac{Area_{extrm}(\gamma_A)}{G_N^{d+1}}$$

Codimension 2 surface

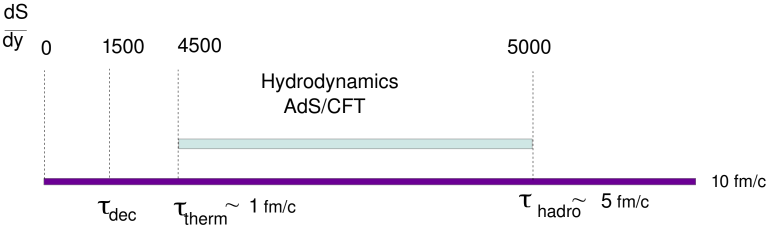
Homology condition

$\gamma_A \sim A$

$\exists$ bulk region r s.t. $\delta r = \gamma_A \cup A$

Holographic Thermalization ,Balasubramanian et al
Entanglement Entropy in AdS-RN-Vaidya

Motivation : QGP



- Goal: study QGP at times $\tau_{deco} < \tau < \tau_{therm}$ using AdS/CFT

Holographic Thermalization, Balasubramanian et al
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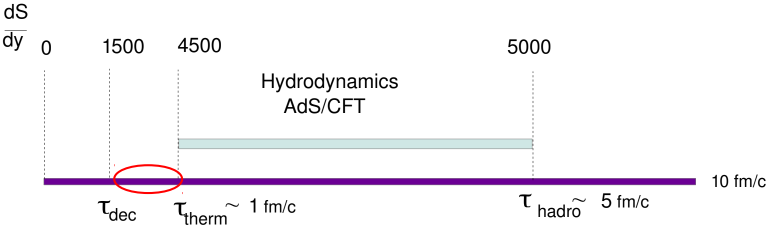
Motivation : QGP



- Goal: study QGP at times $\tau_{deco} < \tau < \tau_{therm}$ using AdS/CFT

Balasubramanian et al

Use Vaidya model (null dust) to calculate holographic thermalization time

$$dS^2 = \frac{1}{z^2}[-(1 - m(v)z^d)dv^2 - 2dzdv + d\bar{x}^2]$$

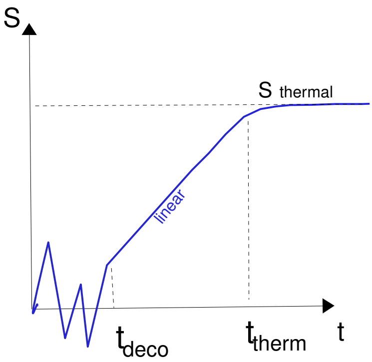
$m(v)$ arbitrarty

Example : $m(v) = \frac{M}{2}(1 + \tanh(\frac{v}{v_0})$

Interpolates between AdS and AdS-Schwarzschild

Signals:

- $\tau_{\mathrm{therm}} \lesssim .7 - 1\mathrm{fm}/c$
- Linear growth of the entropy



from Müller and Schäfer 1110.2378

Introducing a chemical potential

- Need bckgd that interpolates between AdS and AdS-Reissner-Nordström black hole
- Solution to Einstein-Maxwell gravity + Charged null dust

$$R_{\mu\nu} - \frac{1}{2}(R - 2\Lambda)g_{\mu\nu} - g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta} + \frac{1}{4}g_{\mu\nu}F^2 = 2\kappa T^{ext}_{\mu\nu}$$

$$\partial_\alpha[-\sqrt{-g}g^{\mu\alpha}g^{\nu\sigma}F_{\mu\nu}] = J^\sigma_{ext}$$

Ansatz,

$$ds^2 = \frac{1}{z^2}\left(-f(z,v)dv^2 - 2dvdz + d\bar{x}^2\right)$$

Explicit form of $f(z,v)$, $T_{\mu\nu}, J^\sigma$ depend on d

▸ $d = 2, \quad d = 3$

In $d = 4$,

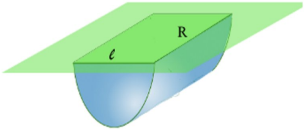
$$ds^2 = \frac{1}{z^2}\left(-f(z,v)dv^2 - 2dvdz + d\vec{x}^2\right)\ , \quad A_v = q(v)z^2\ ,$$
$$f(z,v) = 1 - m(v)z^4 + \frac{2q(v)^2}{3}z^6$$
$$\kappa J^{\mu}_{\mathrm{ext}} = 2\frac{dq}{dv}\delta^{\mu z}\ , \quad 2\kappa T^{\mathrm{ext}}_{\mu\nu} = z^3\left(\frac{3}{2}\frac{dm}{dv} - 2\,z^2 q(v)\frac{dq}{dv}\right)\delta_{\mu v}\delta_{\mu v}\ .$$

Any $m(v), q(v)$.
$d = 4$, take,

$$m(v) = \frac{M}{2}\left(1 + \tanh\frac{v}{v_0}\right) \qquad q^2(v) = Q^2 m(v)^{3/2}$$

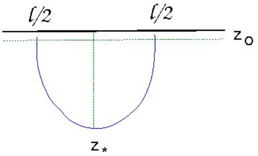
Calculation of EE, rectangular region

Extremize



$$\mathrm{Vol} = \int_{\ell/2}^{\ell/2} \frac{dx}{z^3}\left(1 - f(z,v)v'^2 - 2v'z'\right)^{1/2}$$

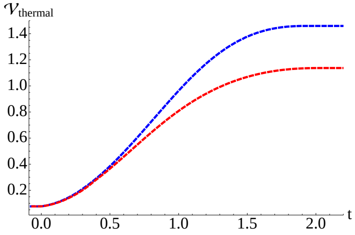
Boundary conditions

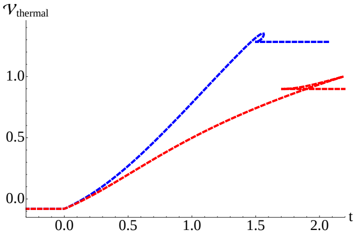


$$z(0) = z_* \quad z'(0) = 0$$
$$v(0) = v_* \quad v'(0) = 0$$

(z_*, v_*) determine a solution $z(x), v(x)$. Then

$$z(\ell/2) = z_0$$
$$v(\ell/2) = t$$



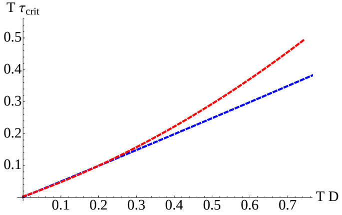


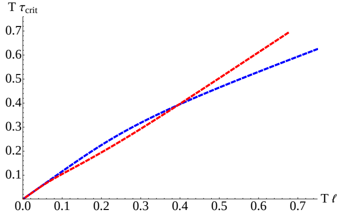
HEE , d=4, circular and rectangular regions, $\chi \approx 0.001$,$\chi \approx 1.1$ where

$$\chi = \mu/T$$

Thermalization time

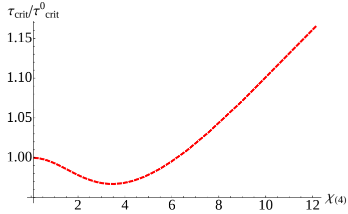
τ_{critical}, time for the geodesic to graze the shell at $v = 0$,
$\tau_{\text{crit}} = \int_{z0}^{z_*} \frac{dz}{f(z)}$





- Thermalization time from UV to IR

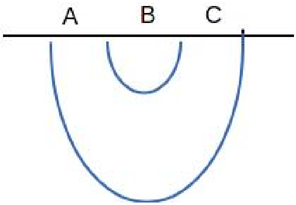
$\tau_{critical}$ AS FUNCTION OF μ/T

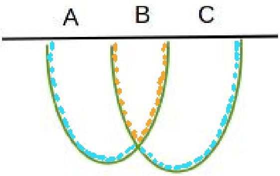


- $4\pi T\ell = 0.5$
- Non monotonic behavior for $T\ell << 1$
- Monotonic for $T\ell >> 1$

Strong Subadditivity and Null Energy Condition

Holographic Entanglement Entropy (HEE) in static backgrounds always satisfies SSA.





$$S(A \cup B \cup C) + S(B) \leq S(A \cup B) + S(B \cup C)$$

Proof does not go through in time dependent backgrounds

SSA is a fundamental property. What does it correspond in the bulk? What is violated in the bulk when SSA does not hold?

- If $S(l)$ is a concave function of l then SSA is satisfied. Callan, He, Headrick; Allais, Tonni.

Experimental :

$T_{\mu\nu} = \frac{dm}{dv} > 0 \rightarrow S(l)$ is concave and SSA is OK

$T_{\mu\nu} = \frac{dm}{dv} < 0 \rightarrow S(l)$ is NOT concave, SSA is violated

Null Energy Condition requires $T_{\mu\nu} > 0$

SSA $\Leftrightarrow$ NEC ?

Proof for time dep. background : A. Wall Proof does not apply to spaces with event horizons

AdS-RN-Vaidya

$$ds^2 = \frac{1}{z^2}(-f(z,v)dv - 2dzdv + dx^2)$$

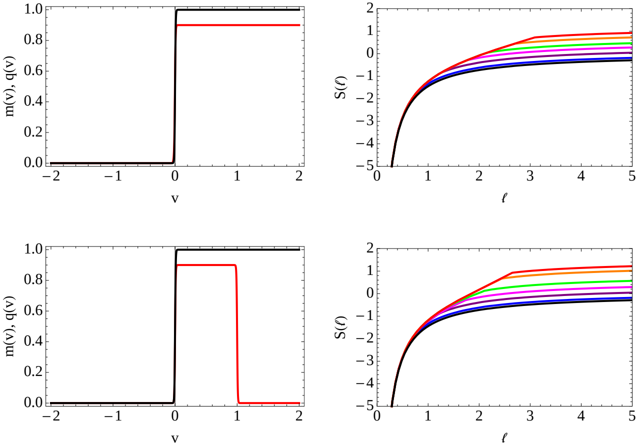
$$f(z,v) = 1 - m(v)z^3 + \frac{q(v)^2}{2}z^4 \qquad \Rightarrow T_{vv} \sim z^3(\frac{dm}{dv} - q(v)\frac{dq}{dv}z)$$

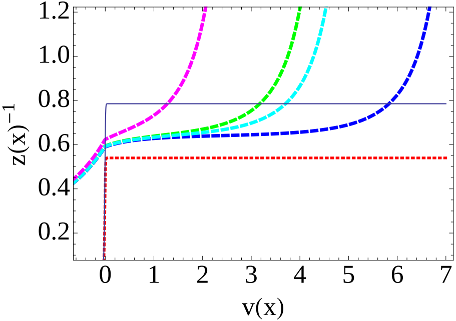
$\exists$ critical surface $z_c = \frac{dm/dv}{q\ dq/dv}$

AdS-RN-Vaidya: non-trivial test of SSA $\leftrightarrow$ NEC

Backgrounds that respect SSA

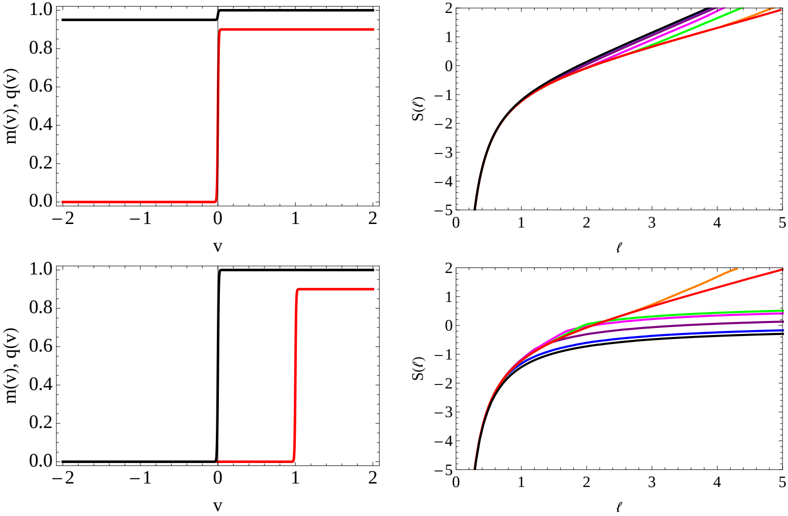
In graphs $m(v)$, $q(v)$ red curve is charge, black curve is mass. Different combinations of $\tanh(v)$.





Intriguing limiting surface

BACKGROUNDS THAT VIOLATE SSA



- SSA constrains geometry
- When SSA is OK
    1. Crit. sfce. is behind the apparent horizon $z_c > z_{ah}$.
    2. Diagnostic function,

$$\mathrm{TD} = -\frac{df}{dz}\Big|_{z=ah} > 0$$

COLLAPSE OF NEUTRAL SCALAR, WEAK FIELD LIMIT

(WORK IN PROGRESS)

Motivation: how far can Vaidya get us?

- Initial state:black hole with M_i, Q
- Poincare patch
- Collapse of non-minimally coupled neutral scalar
- Final state: M_f, Q
- Vaidya to first order en ϵ

Non- minimally coupled massless scalar, Einstein-Maxwell gravity, $\Lambda < 0$, $D = 4$

$$S = \int d^4x\sqrt{g}(R + \Lambda + -\frac{1}{4}h(\phi)F_{\mu\nu}F^{\mu\nu}\frac{1}{2}(\partial\phi)^2)$$

Ansatz,

$$ds^2 = -2dzdv - g(z,v)dv^2 + f(z,v)^2 d\bar{x}^2$$
$$\phi(z,v)$$

(z,v) Eddington-Finkelstein coordinates

AdS-RN initial condition,

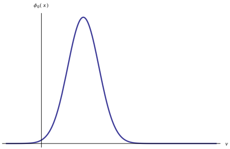
$$\phi(z,v) = 0, \qquad v < 0$$

$$f(z,v) = \frac{1}{z}, \qquad v < 0$$

$$g(z,v) = \frac{1}{z^2} + M_i \epsilon_1^2 z^3 + \frac{Q^2}{2}\epsilon_1^4 z^4, \qquad v < 0$$

at the boundary,

$$\lim_{z\longrightarrow 0} \phi(z,v) = \begin{cases} 0 & v < 0 \\ \phi_0(v)\ \epsilon & 0 < v < \delta t \\ 0 & v > 0 \end{cases}$$



Dilaton pulse, coupling quench

$$f(z,v) = \sum_{n=0}^{\infty} f_n(z,v)\ \epsilon^n$$

$$g(z,v) = \sum_{n=0}^{\infty} g_n(z,v)\ \epsilon^n$$

$$\Phi(z,v) = \sum_{n=0}^{\infty} \phi_n(z,v)\ \epsilon^n$$

$$A(z,v) = \sum_{n=0}^{\infty} A_n(z,v)\ \epsilon^n$$

$z \to 0$ boundary conditions,

$$
\begin{aligned}
f(z,v) &= \frac{1}{z}(1+\mathcal{O}(z^2)) \\
g(z,v)&\frac{1}{z^2}(1+\mathcal{O}(z^2)) \\
\Phi(z,v) &= \phi_0(v)+\mathcal{O}(z) \\
A(z,v) &= \mu+\mathcal{O}(z)
\end{aligned}
$$

initial conditions $0 < v$,

$$
\begin{aligned}
f_0(z,v) &= \frac{1}{z} \\
g_0(z,v) &= \frac{1}{z^2} - M\epsilon_1^2 z + \frac{Q^2}{2}\epsilon_1^4 z^2 \\
\Phi_0(z,v) &= 0 \\
A_0(z,v) &= \mu_i \epsilon_1 - Q\epsilon_1 z
\end{aligned}
$$

One more thing : $\epsilon \sim \epsilon_1$

Solve order by order in ϵ,

$$g(z,v) = \frac{1}{z^2} - M_i z\epsilon_1^2 + \frac{Q^2}{2}z^2\epsilon_1^4 + \left(zC_2(v) - \frac{3}{4}\phi_0'(v)^2 - \frac{M_i}{12}z^3\phi_0'(v)^2\epsilon_1^2\right)\epsilon^2$$

$$+ \left(zC_4(v) + \frac{z^2}{24}(12P(v)\phi_0'(v) - \phi_0'(v)^4) + \right.$$
$$\left. \frac{z^3}{24}(2C_2(v)\phi_0'(v)^2 - 2\phi_0'(v)P'(v) + 6P(v)\phi_0''(v)\right)\epsilon^4 + \mathcal{O}(\epsilon^6)$$

$$f(z,v) = \frac{1}{z} - \frac{1}{8}z^2\phi_0'(v)\epsilon^2 + \frac{z^3}{384}(-48P(v)\phi_0'(v) + \phi_0'(v)^4)\epsilon^4 + \mathcal{O}(\epsilon^6)$$

$$\Phi(z,v) = (\phi_0(v) + z\phi_0'(v))\epsilon + z^3P(v)\epsilon^3 + \mathcal{O}(\epsilon^5)$$

$$A(z,v) = (Qz+\mu)\epsilon_1^2 + a_1(v)\epsilon + \frac{\epsilon^2\epsilon_1^2 Q}{12}(z^3\phi_0'(v)^2 2(1-2h''(0))$$
$$-6Qz^2\phi_0(v)\phi_0'(v)h''(0)) + zQa_4(v)\epsilon_1^2\epsilon^4 + \ldots$$

where,

$$
\begin{aligned}
C_2(v) &= -\int_{-\infty}^{v} \phi_0'(x)\phi_0'''(x)dx \\
C_4(v) &= \int_{-\infty}^{v} \phi_0'(x)\left(-\phi_0'(x)^3 + \int_{-\infty}^{x} B(y)dy\right) \\
B(x) &= \phi_0'(x)(C_2(x) + \phi_0'(x)\phi_0''(x) \\
P(v) &= -\epsilon_1^2 \int_{-\infty}^{v} (M\phi_0'(x) + B(x))dx \\
a_4(v) &= \dots
\end{aligned}
$$

Final state:

$$g(z,v) = \frac{1}{z^2} - z(M\epsilon_1^2 + C_2(v)\epsilon^2 + C_4(v)\epsilon^4) + \frac{Q^2}{2} z^2 \epsilon_1^4 + \dots$$

$$f(z,v) = \frac{1}{z} + \mathcal{O}(\epsilon^6)$$

$$\Phi(z,v) = z^3 P(v)\epsilon^3 + \mathcal{O}(\epsilon^5)$$

$$A(z,v) = \mu_f + (Qz + \mu)\epsilon_1^2 - z\epsilon_1^2\epsilon^4 Q a_4(v) + \dots$$

$\boxed{\text{metric to order } \epsilon^4 \text{ is Vaidya type}}$

Analytic structure

$$\phi_{2n+1}(z,v)=\sum_{k=0}^{2n-2}\epsilon^{2n+1}\phi_{2n+1}^{k}(v)z^{2n+1-k}$$

For $n\geq 2$, $\phi_{2n+1}^{k}(v)$ are polynomials whose degree increases with n.
Perturbation theory breaks down ($v\sim\frac{\delta t}{\epsilon}^{2/3}$), ressumation.

Initial and final state thermodynamics,

$$T_i = \frac{3\epsilon^{2/3}\tilde{m}_0^{1/3}}{4\pi}\left(1 - \frac{\epsilon^{4/3}c^2}{3\tilde{m}_0^{4/3}h(0)}\right), \quad \mu_i = -\frac{\epsilon^{4/3}c}{\tilde{m}_0^{1/3}h(0)}\left(1 + \frac{\epsilon^{4/3}c^2}{6\tilde{m}_0^{4/3}h(0)}\right)$$

$$|c|_{\text{max}} = \frac{\sqrt{3h(0)}\tilde{m}_0^{2/3}}{\epsilon^{2/3}} \simeq 8.04\,,$$

$$T_f \equiv \frac{1}{T_f} = \frac{3\epsilon^{2/3}\tilde{m}_1^{1/3}}{4\pi}\left(1 - \frac{\epsilon^{4/3}c^2}{3\tilde{m}_1^{4/3}h(0)}\right)$$

$$\mu_f = -\frac{\epsilon^{4/3}c}{\tilde{m}_1^{1/3}h(0)}\left(1 + \frac{\epsilon^{4/3}c^2}{6\tilde{m}_1^{4/3}h(0)}\right)\,,$$

Thermalization

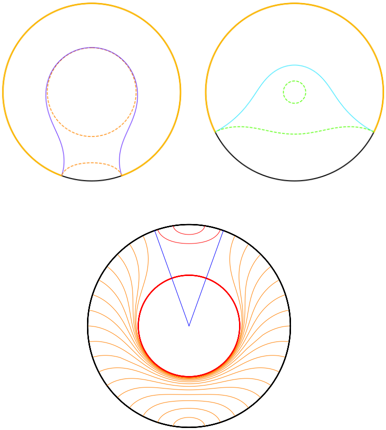
Finite volume effects

- Most of the work done has been done in Poincare patch
- 1312.6887, Hubeny and Maxfield
    - small black holes, entire spacetime is accesible
    - large black holes, exists region that cannot be reached by boundaray anchored geodesics
    - geodesic length is neither monotonic nor contiuos

AdS black hole in global coordinates. BTZ

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2\,d\theta^2 \qquad f(r) = r^2 - r_+^2 \qquad T = \frac{r_+}{2\pi}$$

2 branches of geodesics,



$$\mathfrak{M}_1(\vartheta): \qquad \left\{(r,\theta):\ r=\gamma(\theta,\vartheta,r_+)\equiv r_+\left(1-\frac{\cosh^2(r_+\theta)}{\cosh^2(r_+\vartheta/2)}\right)^{-\frac{1}{2}}\right\},$$

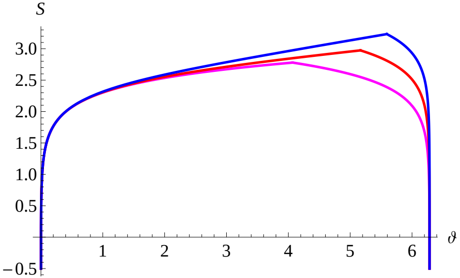
$$\mathfrak{M}_2(\vartheta): \qquad \left\{(r,\theta):\ r=r_+ \ \cup\ r=\gamma(\theta,2\pi-\vartheta,r_+)\right\}.$$

$$S_A=\min\left\{\frac{\mathrm{Area}(\mathfrak{M}_1)}{4G_N^{(3)}},\frac{\mathrm{Area}(\mathfrak{M}_2)}{4G_N^{(3)}}\right\}$$

θ^{χ}_{∞}: exchange of dominance,

$$\theta^{\chi}_{\infty}(r_+) = \frac{1}{r_+}\coth^{-1}(2\coth(\pi r_+) - 1)$$

Entanglement entropy: continuous but not differentiable

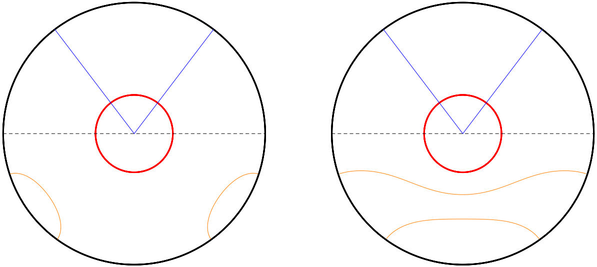


Time evolution of EE in global AdS, much richer arXiv:1312.6887
Hubeny, Maxfield
Time evolution of mutual information?

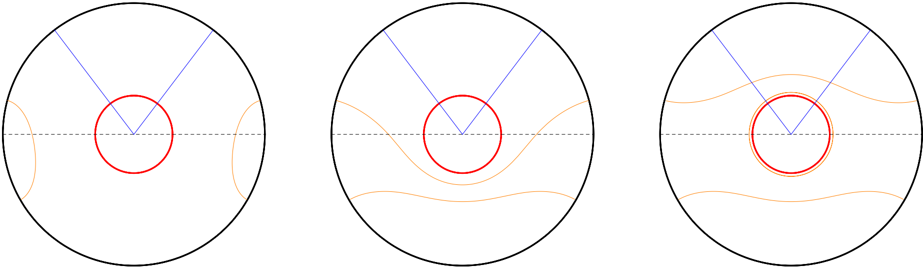
$$I(A,B) = S_A + S_B - S_{A\cup B}$$

First static:

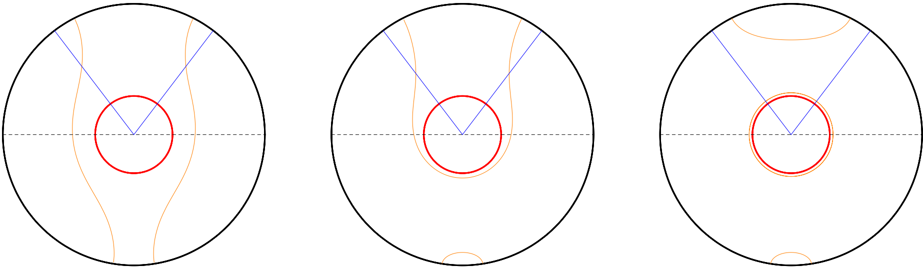
$$S_{A\cup B}?$$



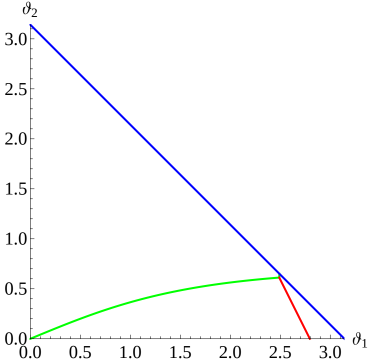
Possible minimal surfaces for region $A \cup B$ for $2\vartheta_1 + \vartheta_2 \leq \pi$. The entropies of these configurations are $2S(\vartheta_1)$ and $S(2\vartheta_1 + \vartheta_2) + S(\vartheta_2)$, respectively. All the surfaces belong to the family $\mathfrak{M}_1$.

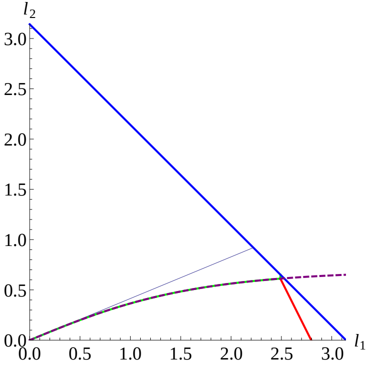


Possible minimal surfaces for region $A \cup B$ for $\pi \leq 2\vartheta_1 + \vartheta_2 \leq \vartheta^x$. The entropies of these configurations are $2S(\vartheta_1)$, $S(2\vartheta_1 + \vartheta_2) + S(\vartheta_2)$ and $S(\vartheta_2) + S(\tilde{\vartheta}_2) + S_H$, from left to right, respectively.



Possible minimal surfaces for region $A \cup B$ for $\vartheta^x \leq 2\vartheta_1 + \vartheta_2 \leq 2\pi$. For this range of parameters, the third option to the right will replace the second one, given that the disconnected surface will have less area and hence $S(2\vartheta_1 + \vartheta_2)$ must be computed from $\mathfrak{M}_2$ solution.


ϑ_2
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.0
0.5
1.0
1.5
2.0
2.5
3.0
ϑ_1



Time evolution?

Conclusions and Future Directions

- Charge and time dependence, interesting arena for many non-trivial checks of HEE
  - Non-monotonic behavior of thermalization time
  - SSA $\leftrightarrow$ NEC
- Weak field limit collapse of a scalar; thermal quench with charge density
- Time evolution of MI, phase structure
- To do:
  - charged scalar collapse, weak field limit, small phase and small norm
  - charged scalar collapse or thermal quench in global AdS
  - full collapse of charged scalar –numerical
  - time evolution of mutual information in global
  - time dependent higher derivatives

Concavity of $S(l) \leftrightarrow$ SSA

Let $a,\ b\ c$ be the lengths of $A,\ B,\ C$ (so $S(A) = s(a),\ S(A \cup B) = s(a+b)$, etc). Let $y = \frac{c}{a+c}$,

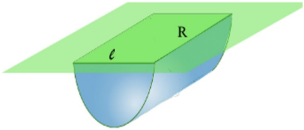
$$a + b = y\ b + (1-y)(a+b+c) \qquad b + c = (1-y)b + y(a+b+c)$$

If s is concave, since $0 < y < 1$

$$s(a+b) \geq ys(b) + (1-y)s(a+b+c) \qquad s(b+c) \geq (1-y)s(b) + ys(a+$$

$$\rightarrow s(a+b) + s(b+c) \geq s(a+b+c) + s(b)$$

Calculation of EE, rectangular region



$$\mathrm{Vol} = \int_{\ell/2}^{\ell/2} \frac{dx}{z^3}\left(1 - f(z,v)v'^2 - 2v'z'\right)^{1/2}$$

x indep. $\quad 1 - f(z,v)v'^2 - 2v'z' = \left(\frac{z_*}{z}\right)^6$

eom 1 $\quad z'' + f(z,v) + \partial_z f z' v' + \frac{v'^2}{2}\partial_v f = 0$

eom 2 $\quad zv'' + 6v'z' - 3 + v'^2\left(3f - \frac{1}{2}z\partial_v f\right) = 0$

In $d = 2$,

$$ds^2 = \frac{L^2}{z^2}\left(-f(z,v)dv^2 - 2dvdz + dx^2\right)\ , \quad A_v = q(v)\log z\ ,$$
$$f(z,v) = 1 - m(v)z^2 + \frac{q(v)^2}{L^2}z^2\log z\ , \quad \Lambda = -\frac{1}{L^2}\ ,$$
$$2\kappa T^{\text{ext}}_{\mu\nu} = \frac{1}{2}z\left(\frac{dm}{dv} - \frac{2}{L^2}\log(z)q(v)\frac{dq}{dv}\right)\delta_{\mu v}\delta_{\nu v}\ , \quad \kappa J^{\mu}_{\text{ext}} = \frac{1}{L}\frac{dq}{dv}\delta^{\mu z}$$

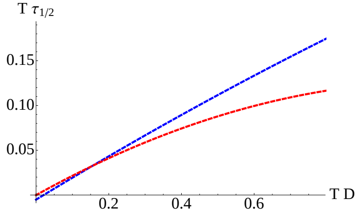
In $d = 3$,

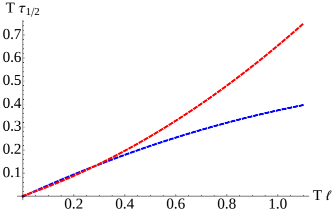
$$ds^2 = \frac{L^2}{z^2}\left(-f(z,v)dv^2 - 2dvdz + d\vec{x}^2\right) \ , \quad A_v = q(v)z \ ,$$
$$f(z,v) = 1 - m(v)z^3 + \frac{q(v)^2}{2L^2}z^4 \ , \quad \Lambda = -\frac{3}{L^2} \ ,$$
$$\kappa J^{\mu}_{ext} = \frac{dq}{dv}\delta^{\mu z} \ , \quad 2\kappa T^{ext}_{\mu\nu} = z^2\left[\frac{dm}{dv} - \frac{z}{L^2}q(v)\frac{dq}{dv}\right]\delta_{\mu v}\delta_{\nu v}$$

◂ $d = 4$

THERMALIZATION : RESULTS

$\tau_{1/2}$, time for the entanglement entropy to reach 1/2 of its maximum value





Apparent horizon, boundary of a trapped surface

- S : codimension 2 surface embeded in a constant t slice
- Incoming and outgoing null geodesics normal to S,

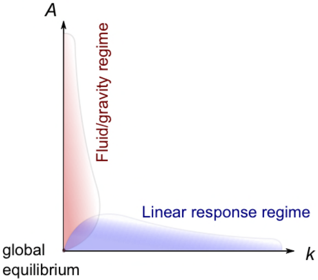
$$N_{in} = \partial_z \qquad\qquad N_{out} = \partial_v + \frac{1}{2} g_{vv} \partial_z$$

- S is an apparent horizon if the Θ expansion of outgoing null geodesics is zero, $\Theta_{out} = 0$.

$$\begin{aligned} \Theta^{out} &= \tilde{g}^{\mu\nu} \nabla_\mu N^{out}_\nu \\ &= (g^{\mu\nu} + N^{\mu}{}_{in} N^{\nu}_{out} + N^{\mu}_{out} N^{\nu}_{in}) \nabla_\mu N^{out}_\nu \end{aligned}$$

For $\mu = 0$, $\Theta^{out} = \frac{d-1}{2}(1/z - m(v) z^{d-1}) \Rightarrow z_h = m(v)^{-1/d}$

- Long wave approx. Fluid/gravity duality



$$k = \ell_{mfp}/L$$

graph from Hubeny and Rangamani, 1006.3675