Relativistic Runge-Lenz Vector

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Motivation

- `N=4 SYM: the harmonic oscillator of the 21st century?'
- One is generally interested in two types of theories:

  Theories which precisely describe Nature

  Approximate models which we can exactly solve

CENTRAL IDEA

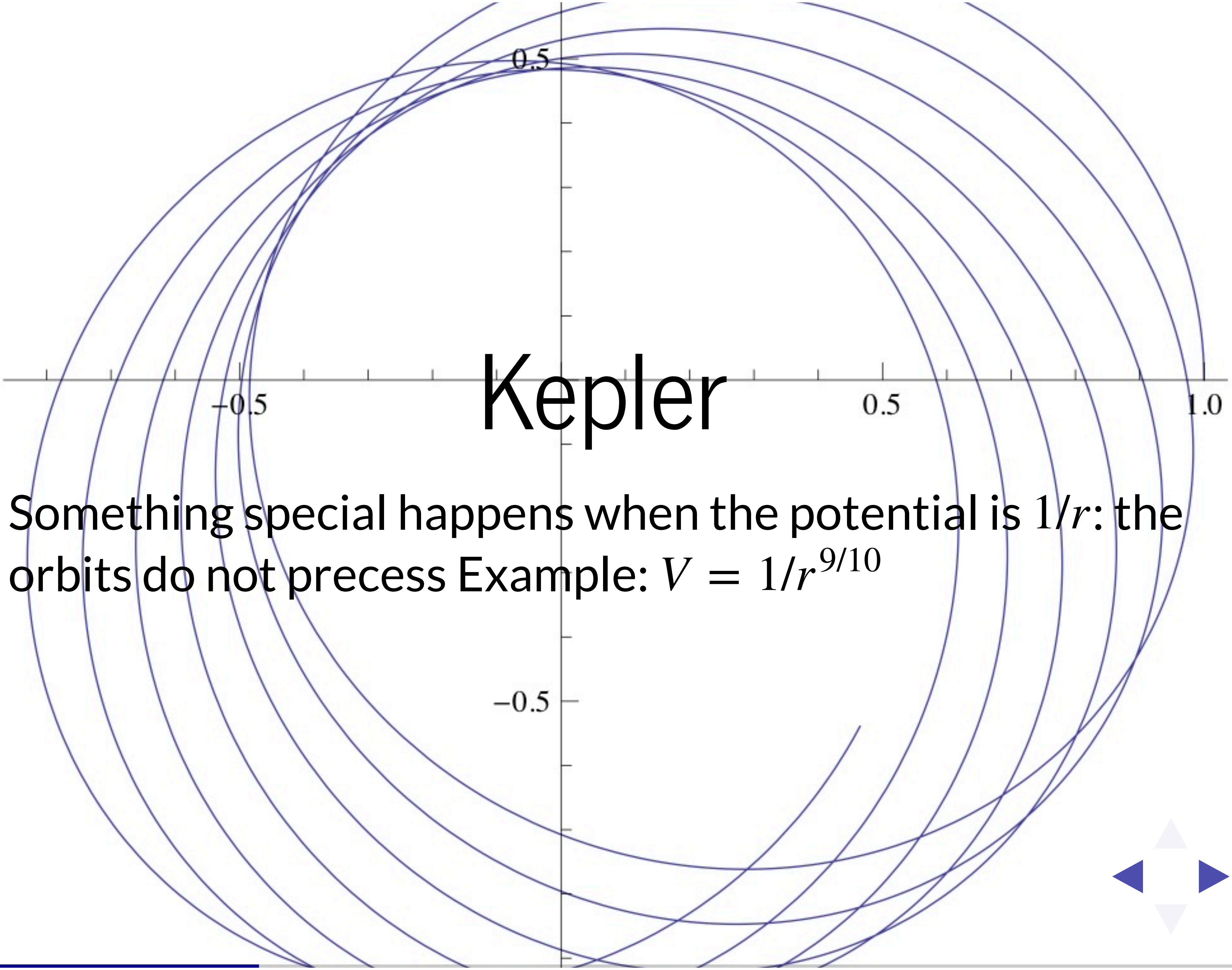
- N=4 SYM: the *Hydrogen atom* of the 21st century?
- In four dimensions, a unique nontrivial quantum field theory in which we can calculate scattering amplitudes exactly!!! → Integrable
- There is a symmetry that makes it possible?

Kepler

- Consider the classical two-body problem with a 1/r potential We can go to a center-of-mass frame; four conserved quantities are apparent: angular momentum $\vec{J}$ and energy
- These basically fix the dynamics, for example the motion takes place in a plane, ...

Kepler

- Something special happens when the potential is $1/r$: the orbits do not precess Example: $V = 1/r^{9/10}$



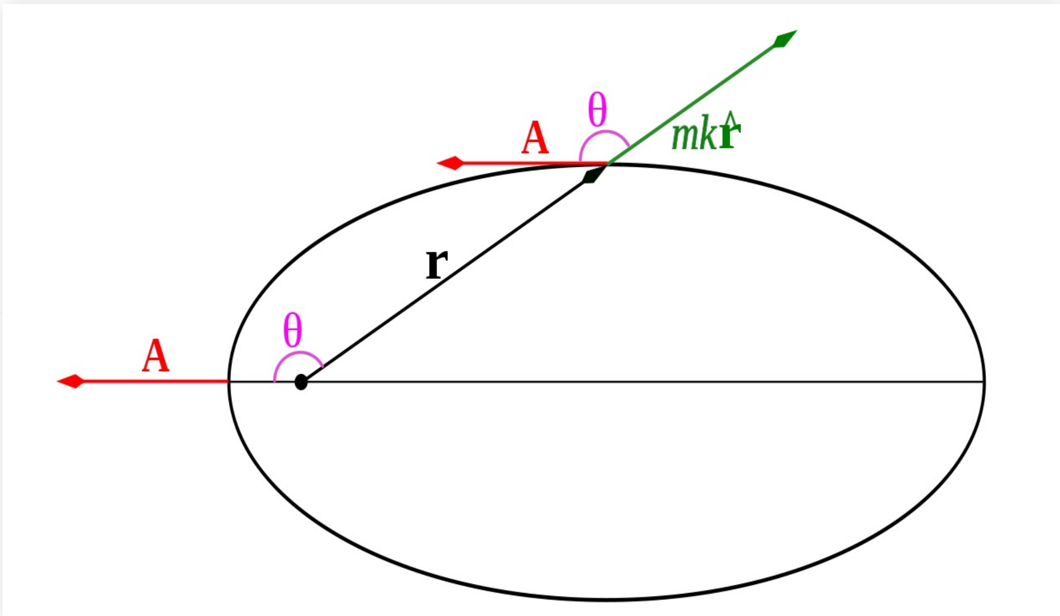
Kepler

For $V \sim -\frac{1}{r}$ the system possess an additional, non-obvious conserved vector:

$$\vec{M} = \frac{1}{m}\vec{p} \times \vec{J} - \frac{\vec{x}}{|x|}$$

(« Laplace-Runge-Lenz » vector)

- Quantum mechanically, the Laplace-Runge-Lenz vector is still conserved
- It explains the well-known degeneracy of the excited states of the Hydrogen atom (this was quickly pointed out by Pauli in the early days of QM)
- It points in the direction of the eccentricity, preventing it from precessing



Is there a fully consistent, relativistic quantum field theory, in which the Runge-Lenz vector is conserved?

YES !!!!!

N=4 SYM: the *Hydrogen atom* of the 21st century

$\mathcal{N} = 4$ SYM

- Amplitudes in N=4 SYM have a symmetry. The symmetry is non-obvious, but there is a *conformal symmetry in momentum space* S0(6)
- This symmetry (in amplitudes with massive legs) reduces to SO(4) symmetry and contains the usual SO(3) subgroup. What are the remaining three generators? The Runge-Lenz vector!

The SO(2,4) dual conformal symmetry in the massless sector is at the heart of the Wilson loop/ amplitude duality, of the integrability of the N=4 theory, and of other recent developments.

- The N=4 theory comes with a moduli space of vacua, parametrized by 6 adjoint scalar fields
- We can give them the vev's we want. For example by breaking: SU(Nc)→U(1)xSU(Nc-1) we will get U(1) ``photons'' coupled to massive W bosons.

Lets take the minimal example (the scalar sector of the bosonic SYM theory)
Lagrangian

$$\mathcal{L} = \mathrm{Tr}\left\{ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} \sum_{i=1}^{6} D_\mu \Phi_i D^\mu \Phi_i - \frac{g^2}{4} \sum_{i,j=1}^{6} [\Phi_i, \Phi_j]^2 \right\}$$

Here the six scalar fields are $N \times N$ traceless hermitian matrices in the adjoint representation of SU(N).
Equations of motion

$$D_\mu F^{\nu\mu} = ig \sum_{i=1}^{6} [\Phi_i, D^\nu \Phi_i],$$

$$D_\mu D^\mu \Phi_i = g^2 \sum_{j=1}^{6} [\Phi_j, [\Phi_j, \Phi_i]].$$

Giving a vev to $\Phi_1 = \Phi_1^0 + v$ and taking SU(2) for simplicity we can make the explicit Higgsing

After some identifications ($\Phi_2^- = \phi, \Phi_2^+ = 0, A_\mu^3 = A_\mu$), the equation of motion for the scalar fields are

$$\partial_\mu \partial^\mu \phi - 2iA^\mu \partial_\mu \phi - A^\mu A_\mu \phi - \left(m - \frac{\alpha}{r}\right)^2 \phi = 0,$$

where the vacuum expectation value of Φ_1 is the mass m of the scalar field, and we fixed the field Φ_1^0 to be $-\alpha/r$.

non-minimal coupling $m \to (m - \alpha/r)$ induced by the Higgs mechanism

Taking the particular solution for the equations of motion of the A_μ

$$A_\mu = (\alpha/r, \mathbf{0})$$

the quadratic term coming from the minimal coupling (introduced through the covariant derivative), just cancel with the non- minimal quadratic term, enhancing the symmetry of the resulting field theory from SO(3) to SO(4). The scalar field theory come from N=4 SYM and have the SO(4) Runge-Lenz symmetry!!!!

Consistency checked !

Particle model:

Consider the standar minimal coupling. Lagrangian:

$$S = -mc \int d\tau \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} + \frac{1}{c} \int d\tau A_\mu \frac{dx^\mu}{d\tau},$$

Eq. of Motion,

$$\eta^{\mu\nu} \left(p_\mu - \frac{1}{c} A_\mu \right) \left(p_\nu - \frac{1}{c} A_\nu \right) + m^2 c^2 = 0,$$

Coulomb field $A_0 = \alpha/r, A_i = 0$, Quantization

$$\left[\left(-i\partial_t - \frac{\alpha}{r} \right)^2 + \nabla^2 - m^2 \right] \phi(x) = 0.$$

Energy levels (relativistic effects, degenerancy broken)

$$E_{\ell,n} = m \left[1 + \left(\frac{\alpha}{n + \sqrt{(\ell + 1/2)^2 - \alpha^2} - (\ell + 1/2)} \right) \right]^{-1/2},$$

Non- minimal coupling model, Lagrangian:

$$S=\int d\tau\left[-m\sqrt{-\left(1-\frac{\alpha}{rm}\right)^{2}\eta_{\mu\nu}\dot{x}^{\nu}\dot{x}^{\mu}}+\frac{\alpha}{r}\dot{x}^{0}\right].$$

Eq. of motion

$$p_0^2-\frac{2\alpha}{r}p_0-\vec{p}^{\,2}-m^2+\frac{2\alpha m}{r}=0.$$

Quantization:

$$\left[\left(-i\partial_t-\frac{\alpha}{r}\right)^{2}+\nabla^2-(m-\alpha/r)^2\right]\phi(x)=0.$$

Spectra

$$E_n=m\left(1-\frac{2\alpha^2}{n^2+\alpha^2}\right).$$

Hydrogen atom spectra → RELATIVISTIC RUNGE LENZ!→ Integrability

Relativistic Runge-Lenz vector

The relativistic generalization of the Runge-Lenz vector is then

$$\vec{K} = \vec{p} \times \vec{L} - (E_0 + E)\,\frac{\alpha \vec{x}}{r}.$$

NR limit $(E + E_0) \to 2m, (E - E_0) \to E_{NR}, E_0$ is the rest Energy

$$\vec{K}_{NR} = \vec{p} \times \vec{L} - 2\alpha m \frac{\vec{x}}{r},$$

Relativistic Runge-Lenz Symmetry

Relativistic Runge-Lenz symmetry (Noether Theorem)

$$\delta r^i = 2(\epsilon \cdot r)p_i - r^i(\epsilon \cdot p) - (r \cdot p)\epsilon^i,$$

$$\delta p_i = -(\vec{p})^2\epsilon_i + (\vec{p} \cdot \vec{\epsilon})p_i - \alpha\,(E + E_0)\,(\frac{\vec{\epsilon} \cdot \vec{x}}{r^3}x_i - \frac{\epsilon_i}{r}).$$

Relation with the dual conformal symmetry?

Orbit from RL Vector

From $\vec{K} \cdot \vec{x}$

$$\frac{L^2}{r\left(E_0 + E\right)\alpha} = \frac{K\cos\theta}{\alpha\left(E_0 + E\right)} + 1,$$

that is, the equation of a conic with eccentricity

$$\varepsilon = \frac{K}{\alpha\left(E_0 + E\right)}.$$

Here, we notice that under the substitution $(E_0 + E) \to 2m$ we recover the NR result

Symmetry Algebra

The corresponding relativistic algebra closes as

$$[L_i, L_j] = i\varepsilon_{ijk} L_k,$$
$$[A_i, J_j] = i\varepsilon_{ijk} A_k,$$
$$[A_i, A_j] = -4i \left(\frac{E-m}{E+m} \right) \varepsilon_{ijk} L_k,$$

We can also see the algebra SO(4)= SO(3)$\times$ SO(3)

The correct relativistic spectrum can be reconstructed from this algebra just as the NR case (Pauli)

Relativistic Schroediger Equation

We define the radial part of the relativistic Schroedinger equation (RSE) in d dimensions as

$$\left(-\frac{1}{(E+m)}\left(\frac{d^2}{dr^2}+\frac{(d-1)}{r}\frac{d}{dr}-\frac{\ell(\ell+d-2)}{r^2}\right)+2V(r)-(E-m)\right)R(r)=0.$$

that follows from the KG Equation

$$(E+E_0)\left(E-E_0-2V-\frac{1}{(E+E_0)}(p_r^2+\ell^2/r^2)\right)=0,$$

Dual RSE

We can relate to equivalent descriptions of RSE mapping potentials of the form

$$V(r) = kr^{\beta} \quad \rightarrow \quad \mathcal{V} = -\frac{1}{2}(E-m)\left(\frac{2}{\beta+2}\right)^2 \rho^{-\frac{2\beta}{\beta+2}}.$$

where $r = \rho^{2/(\beta+2)}$

Dictionary:

$$D = \frac{2(\beta+d)}{\beta+2}, \quad \mathcal{L} = \frac{2}{\beta+2}\ell.$$

$$\mathcal{E} - m = -2(\frac{2}{\beta+2})^2 k \quad E + m \rightarrow \mathcal{E} + m,$$

- Starting from the hydrogen atom $\beta = -1$ in $d = 3$ we can obtain the isotropic harmonic oscillator in $D = 4$ and angular momenta $\mathcal{L} = 2\ell$.
- The energy spectrum of the relativistic hydrogen atom is
$$(2n + 2\ell + 2)\sqrt{m - E} - 2\alpha\sqrt{E + m} = 0,$$
and from here we have
$$E = m(1 - \frac{2\alpha^2}{N^2 + \alpha^2}),$$
with $N = n + \ell + 1$.
- The new potential is
$$\mathcal{V} = -2(E - m)\rho^2,$$
- According to our dictionary the prediction for the harmonic oscillator spectra is
$$\mathcal{E} - m = 2(4n + 2\mathcal{L} + 4)\sqrt{\frac{E - m}{\mathcal{E} + m}}.$$

N=4 SYM is the harmonic oscillator and the hydrogen atom of the 21st century !