Matter contributions to Holographic Entanglement Entropy

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Plan for the talk

1. Introduction: Generalized gravitational entropy.
2. The set up: Lovelock gravity conformally coupled to matter.
3. Calculation of the holographic entanglement entropy.
4. Results.
5. Conclusions and future work.

Motivations

- Entanglement entropy is a quantity of great interest in quantum field theory. The holographic prescription for it, provides us with a relation between the entanglement properties of a quantum theory and geometric quantities in its gravity dual.
- It is well understood how to calculate this quantity in Einstein gravity, and even in higher order derivative theories, such as Lovelock gravity. However, there are very few examples of how this works when we couple the theory non-minimally with some matter field.
- We would like to find the holographic dual for a particular solution to a theory which contains higher order terms as well as non minimal couplings to a scalar field. This solution is a 5-dimensional spherical hairy black hole. We hope the entanglement entropy would help us to characterize the properties of this dual.

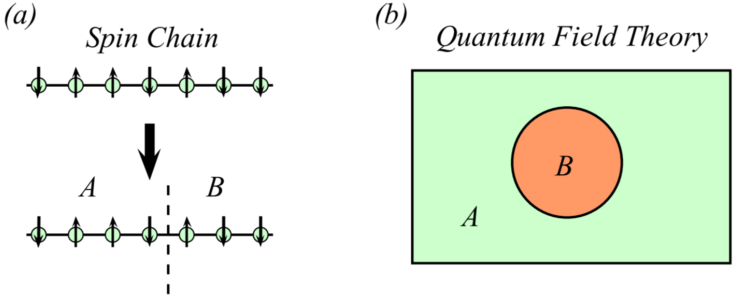
Entanglement entropy (EE)

The Von Neumann entropy asociated to the *reduced* density matrix:

$$\rho_A = Tr_B\ \rho \qquad \textit{for} \quad \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \tag{1}$$

is the entanglement entropy of the subsystem A:

$$S_A = -Tr(\rho_A \log \rho_A) \tag{2}$$



Replica method

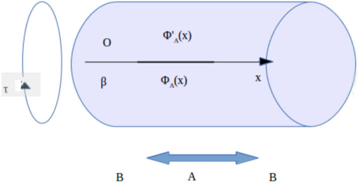
The entanglement entropy satisfies the following identity:

$$S_A = \lim_{n\to 1} S_n \equiv \lim_{n\to 1} \frac{Tr[\rho_A]^n - 1}{1-n} = -\frac{\partial}{\partial n} \log Tr[\rho_A^n]|_{n=1} \qquad (3)$$

where the S_n are the Renyi entropies.

For $n = 1$, we can calculate $Tr[\rho_A]$ using the following:

$$Tr[\rho_A] = Z^{-1} \int_{\tau=-\infty}^{\tau=\infty} \mathcal{D}\phi e^{-S_E[\phi(x)]}$$



For arbitrary n, we can construct n copies of the manifold and the fields, meaning:

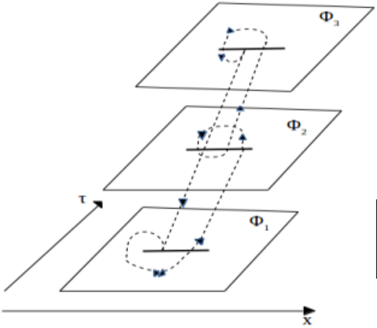
$$[\rho_A^n] = [\rho_A]_{\phi_{1+}\phi_{1-}}[\rho_A]_{\phi_{2+}\phi_{2-}} ... [\rho_A]_{\phi_{n+}\phi_{n-}} \qquad (4)$$

Replica method

Joining these copies by means of the boundary conditions:

$$\begin{aligned} \phi_j(0^+, x) = \phi_{j+1}(0^-, x) \quad x \in A \\ \phi_j(0^+, x) = \phi_j(0^-, x) \quad x \in B \end{aligned} \tag{5}$$

We can interpret this as integrating over a n-sheeted Riemannian manifold, $\mathcal{M}_n$.



$$Tr_A \ \rho_A^n = \frac{1}{(Z)^n} \int_{(t,x)\in\mathcal{M}_n} D\phi \ e^{S_n[\phi]}$$

$$\equiv \frac{Z_n}{(Z)^n}$$

$$\boxed{S_A = -\lim_{n\to 1} \frac{1}{n-1} (\log Z_n - n \log(Z_1)^n)}$$

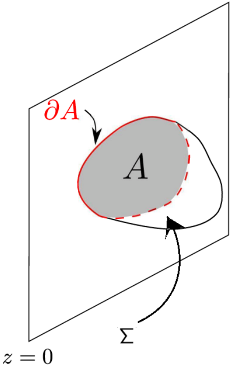
[Calabrese, Cardy]

Holographic entanglement entropy

The AdS/CFT correspondence provides us with a prescription to obtain S_A from the (Einstein) gravity theory:

$$\boxed{S_A = \frac{Area(\Sigma)}{4G} = \frac{1}{4G}\int d^{d-1}x\sqrt{h}} \qquad (6)$$

[Ryu, Takayanagi]



Σ is the codimension 2 surface, homologous to the region A, which minimizes the functional S_A.

- What happens if the gravity theory is not Einstein?

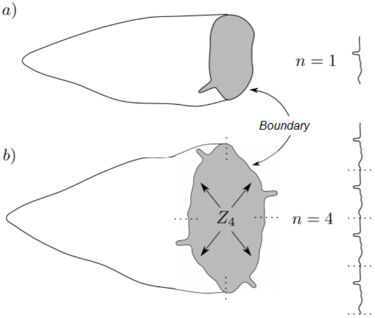
Generalized gravitational entropy

Consider a theory which has a CFT on $\mathcal{M}_n$ as holographic boundary. We call this manifold B_n. Using the AdS/CFT dictionary we can write the corresponding action as:

$$I[B_n] = -\log(Z[\mathcal{M}_n]) \equiv -\log\ Z_n \tag{7}$$

B_n can be constructed by gluing n copies of the original manifold around a periodic coordinate. Taking the quotient with the symmetry group $\mathbb{Z}_n$, we obtain an orbifold $\hat{B}_n$, which has a period 2π (instead of $2\pi n$):

$$\hat{B}_n = \frac{B_n}{\mathbb{Z}_n} \quad \Longrightarrow \quad I[\hat{B}_n] = nI[B_n]$$



Squashed cone method

Then:

$$S_A = \lim_{n\to 1} \frac{n}{n-1}\left(I[\hat{B}_n] - I[\hat{B}_1]\right) = \partial_n I[\hat{B}_n]\big|_{n=1}$$

[Lewkowycz, Maldacena]

Due to the induced conical singularity on the orbifold, we must regularize this geometry:

$$S_A = -n\partial_n[(\ln Z_n - \ln Z_n^{off}) - (\ln Z_n^{off} - \ln Z_1^n)]_{n=1} \qquad (8)$$

where Z_n^{off} corresponds to the manifold which regularizes $\hat{B}_n$, by substituting the tip of the cone with a regular function at $r = a$. The second term vanishes, and we end up with:

$$S_A = \partial_n I[\hat{B}_n]\big|_{n=1} = \partial_\epsilon I[B_n]|_{\epsilon=0} \qquad (9)$$

where $\epsilon = 1 - n^{-1}$.

Squashed cone method

To obtain explicitly the terms in this last derivative, we expand the metric of the cone using ϵ as an expansion parameter, and switching to the normal complex coordinates $(z, \bar{z})$ we have:

$$
\begin{aligned}
ds^2 = e^{2A}[dzd\bar{z} + e^{2A}T(\bar{z}dz - zd\bar{z})^2] + (g_{ij} + 2K_{aij}x^a \\
+ Q_{abij}x^a x^b)dy^i dy^j + 2ie^{2A}(U_i + V_{ai}x^a)(\bar{z}dz - zd\bar{z})dy^i + \ldots,
\end{aligned}
\tag{10}
$$

[Dong]

where

$$
A = -\frac{\epsilon}{2}\log(z\bar{z}) = -\frac{\epsilon}{2}\log(\rho^2 + a^2) \tag{11}
$$

Some of the order one terms constructed with this metric are:

$$
R \sim 4\pi\epsilon e^{-2A}\delta^2(z,\bar{z}), \qquad R_{zz} \sim -\frac{\epsilon}{z}K_z, \qquad R_{\bar{z}\bar{z}} \sim -\frac{\epsilon}{\bar{z}}K_{\bar{z}}, \tag{12}
$$

and

$$
R_{z\bar{z}} \sim \pi\epsilon\delta^2(z,\bar{z}), \qquad \Gamma^z_{zz} \sim -\frac{\epsilon}{z}, \qquad \Gamma^{\bar{z}}_{\bar{z}\bar{z}} \sim -\frac{\epsilon}{\bar{z}} \tag{13}
$$

Example: Einstein Gravity

We start with the Einstein-Hilbert action in 4-dim:

$$I_{EH} = \frac{1}{16\pi G_N} \int d^4x \sqrt{g} R \tag{14}$$

Recall that

$$S_A = \partial_\epsilon ((I_{EH})_n)|_{\epsilon=0},$$

so we must consider only linear terms in ϵ-expansion of the metric. Using also the expression for the order one term in the determinant of the metric $\sqrt{g}$ we have:

$$(I_{EH})_n \sim \frac{4\pi\epsilon}{16\pi G_N} \int d^4x \sqrt{g}\ 2e^{-2A}\delta^2(z,\bar{z})$$

$$= \frac{\epsilon}{4G_N} \int dz d\bar{z} d^2y (\frac{1}{2} e^{2A}\sqrt{h}) 2e^{-2A}\delta^2(z,\bar{z}) = \frac{\epsilon}{4G_N} \int d^2y \sqrt{h}$$

Then:

$$S_A = \boxed{\frac{1}{4G_N} \int d^2y \sqrt{h} = \frac{Area}{4G_N}}$$

Example withn non-minimally coupled matter

- $\phi^2 R$ theory

It is known that a minimal coupling term, such as $\partial_\mu \phi \partial^\mu \phi$ for a scalar field ϕ, do not contribute to S_A [Lewkowycz, Maldacena]. For a non-minimal coupling term we can expand ϕ, as we did for the metric:

$$\phi = \phi^{(o)} + \phi_z z + \phi_{\bar{z}} \bar{z} + \phi_{zz} z^2 + \phi_{\bar{z}\bar{z}} \bar{z}^2 + \phi_{z\bar{z}} (z\bar{z})^{1-\epsilon} + \ldots \quad (15)$$

We can consider a non-minimal coupling of the type:

$$I_{\phi^2 R} = I_{EH} + \lambda \int d^4x \sqrt{g} \phi^2 R$$

Following the same steps as in in the previous example, and taking ϕ at order zero in the expansion, it is easy to check that:

$$S_A = \frac{Area}{4G_N} + \lambda \int d^2x d^{d-2}y \left(\sqrt{g}\right) (\phi^2)^{(0)} 2(2e^{-2A})^2 (\pi e^{2A} \delta^{(2)}(x_1, x_2))$$

$$= \boxed{\frac{1}{4G_N} \int d^{d-2}y \sqrt{h} (1 + 16\pi G_N \phi^2)}$$

Higher order theories: Camps-Dong prescription

What about higher order theories such as Lovelock?

Let us consider a general gravity theory of the form:

$$I = \int d^D x \sqrt{g} \mathcal{L}(R_{\mu\nu\lambda\rho}),$$

Using the squashed cone method, we can obtain the following formula:

$$S_A = 2\pi \int d^{D-2} y \sqrt{h} \Big\{ \overbrace{\frac{\partial \mathcal{L}}{\partial R_{z\bar{z}z\bar{z}}}}^{\text{Wald}} + \overbrace{\sum_\alpha \left(\frac{\partial^2 \mathcal{L}}{\partial R_{zizj} \partial R_{\bar{z}k\bar{z}l}} \right)_\alpha \frac{K_{zij} K_{\bar{z}kl}}{q_\alpha + 1}}^{\text{“Anomaly”}} \Big\},$$

[Camps, Dong]

where K_{aij} is the extrinsic curvature tensor. The second (anomalous) term comes from a quadratic term in ϵ which is promoted to a linear term after integration.

Example: Lovelock gravity

For an integer k we construct the scalar:

$$R^{(k)} = \delta^{a_1 b_1 \dots}_{c_1 d_1 \dots} \delta^{a_k b_k}_{c_k d_k} R_{a_1 b_1}^{\ \ c_1 d_1} \dots R_{a_k b_k}^{\ \ c_k d_k} \tag{16}$$

And the Lovelock action in D dimensions with coupling constants α_k is:

$$I_{Lovelock} = \int_{\mathcal{M}} d^D x \, \sqrt{-g} \sum_{k=0}^{[\frac{D-1}{2}]} \frac{1}{2^k} \alpha_k R^{(k)} + I_B, \tag{17}$$

[Lovelock]

- This is the most general pure gravity theory with vanishing torsion whose fields equations are given by a rank-2 symmetric tensor, and of second order in the metric g.

Applying the Camps-Dong formula to this action, we obtain the entanglement entropy functional:

$$S_A = -2\pi \int d^{D-2} y \sqrt{h} \frac{k}{2^{k-2}} \ \delta^{\alpha_1 \beta_1 \dots \alpha_{k-1} \beta_{k-1}}_{\mu_1 \nu_1 \dots \mu_{k-1} \nu_{k-1}} \mathcal{R}_{\alpha_1 \beta_1}^{\ \ \mu_1 \nu_1} \dots \mathcal{R}_{\alpha_{k-1} \beta_{k-1}}^{\ \ \mu_{k-1} \nu_{k-1}}.$$

[Dong]

Lovelock gravity conformally coupled to matter

We can couple matter to the Lovelock theory, by defining the following tensor:

$$S_{cd}^{\ ab} = \phi^2 R_{cd}^{\ ab} + \frac{4}{s}\phi\ \ \delta^{[a}_{[c}\nabla_{d]}\nabla^{b]}\phi\ + \frac{4(1-s)}{s^2}\delta^{[a}_{[c}\nabla_{d]}\phi\nabla^{b]}\phi - \frac{2}{s^2}\delta^{[a}_{[c}\delta^{b]}_{d]}\nabla_\mu\phi\nabla^\mu\phi, \qquad \text{[Oliva, Ray]} \tag{18}$$

where ϕ is a scalar field, and s is the conformal weight. Considering an additional set of coupling constants β_k, the action reads:

$$I[g,\phi] = \int_{\mathcal{M}} d^D x\ \sqrt{-g}\ \sum_{k=0}^{[\frac{D-1}{2}]} \frac{1}{2^k}\left(\alpha_k R^{(k)} + \beta_k \phi^{m_k} S^{(k)}\right), \tag{19}$$

where $m_k = \frac{2k(1-s)-D}{s}$.

Lovelock gravity conformally coupled to matter

By conformal, we mean that $g_{\mu\nu} \to e^{2\Omega} g_{\mu\nu}$, $\phi \to e^{s\Omega}\phi$ has the only effect of changing ${S_{\mu\nu}}^{\lambda\rho} \to e^{2(s-1)} {S_{\mu\nu}}^{\lambda\rho}$.

The resulting equation of motion for the metric is:

$$T_{ab} = g_{cb} \sum_{k=0}^{[\frac{D-1}{2}]} \frac{\beta_k}{2^{k+1}} \phi^{m_k} \delta^{ca_1 b_1 \ldots a_k b_k}_{ac_1 d_1 \ldots c_k d_k} S^{c_1 d_1}_{a_1 b_1} \ldots S^{c_k d_k}_{a_k b_k}, \tag{20}$$

and for the scalar field:

$$\sum_{k=0}^{[\frac{D-1}{2}]} \frac{(2k-D)\beta_k}{2^k s} \phi^{m_k - 1} \delta^{c_1 d_1 \ldots c_k d_k}_{a_1 b_1 \ldots a_k b_k} S^{c_1 d_1}_{a_1 b_1} \ldots S^{c_k d_k}_{a_k b_k} = 0 \tag{21}$$

In $D = 5$, choosing $s = -1/5$ and setting $\alpha_2 = 0$, the action reads:

$$I[g,\phi] = \int_{\mathcal{M}} d^5x \ \ \sqrt{-g}\Big(\frac{\Lambda}{8\pi G_N} + \frac{R}{16\pi G_N} + b_0 \phi^{25} + b_1 \phi^{13} S$$
$$+ b_2 \phi \big(S_{cdab} S^{cdab} - 4 S_{ab} S^{ab} + S^2 \big) \Big),$$

Black Hole solution

There is an analytical solution to this theory which is valid under the following conditions:

- Static and spherically symmetric
- Arbitrary sign of the cosmological constant
- A set of constraints over the coupling constants

In $D = 5$, and choosing $s = -1/5$ we have:

$$b_2 = \frac{9}{10}\frac{b_1^2}{b_0}, \quad \tilde{Q}_0 = \frac{64\pi G_N}{6}\epsilon b_1\Big(\frac{-18}{5}\frac{b_1}{b_0}\Big)^{3/2}, \quad N = \epsilon\Big(\frac{-18}{5}\frac{b_1}{b_0}\Big)^{1/10}$$

and the solution is:

$$\boxed{ds^2 = -F(r)dt^2 + F^{-1}(r)dr^2 + r^2 d\Sigma_3^2}$$

$$\boxed{F(r) = 1 - \frac{\tilde{M}}{r^2} - \frac{\tilde{Q}_0}{r^3} + \frac{r^2}{l^2}, \qquad \phi(r) = \frac{N}{r^{1/5}}.}$$

HEE functional for Lovelock theory with conformal couplings

The Lagrangian for this theory can be written in the following form:

$$\mathcal{L}(g_{\mu\nu}, \phi) = \mathcal{L}_g({R_{\mu\nu}}^{\rho\lambda}) + \mathcal{L}_m({S_{\mu\nu}}^{\rho\lambda}),$$

where $\mathcal{L}_g$ is the usual gravity part of the action, and $\mathcal{L}_m$ depends linearly on ${R_{\mu\nu}}^{\rho\lambda}$, as well as on ϕ. Schematically, we can write

$${S_{\mu\nu}}^{\rho\lambda} = \phi^2 {R_{\mu\nu}}^{\rho\lambda} + {\Phi_{\mu\nu}}^{\rho\lambda}(\phi)$$

where

$${\Phi_{cd}}^{ab} = \frac{4}{s}\phi\delta^{[a}_{[c}\nabla_{d]}\nabla^{b]}\phi + \frac{4(1-s)}{s^2}\delta^{[a}_{[c}\nabla_{d]}\phi\nabla^{b]}\phi - \frac{2}{s^2}\delta^{[a}_{[c}\delta^{b]}_{d]}\nabla_\mu\phi\nabla^\mu\phi.$$

Applying the Camps-Dong formula, we can calculate the contribution of the matter part to the HEE functional as:

$$\boxed{S_m = 2\pi\int d^{D-2}y\sqrt{h}\,\phi^2\left\{\frac{\partial\mathcal{L}_m}{\partial S_{z\bar{z}z\bar{z}}} + \phi^2\sum_\alpha\left(\frac{\partial^2\mathcal{L}_m}{\partial S_{zizj}\partial S_{\bar{z}k\bar{z}l}}\right)_\alpha\frac{K_{zij}K_{\bar{z}kl}}{q_\alpha+1}\right\}}$$

Calculation of HEE functional by the Camps-Dong formula

The matter part of the Lagrangian for $D=5$ is:

$$\mathcal{L}_m = b_0\phi^{m_0} + b_1\phi^{m_1}S + b_2\phi^{m_2}\left(S^2 - 4S_{\mu\nu}S^{\mu\nu} + S_{\mu\nu\lambda\rho}S^{\mu\nu\lambda\rho}\right)$$

Having this Lagrangian, we can obtain the corresponding contribution to S_A. First we evaluate the derivatives respect to ${S_{\mu\nu}}^{\rho\lambda}$:

$$\frac{\partial \mathcal{L}_m}{\partial S_{z\bar{z}z\bar{z}}} = 2\Big[b_1\phi^{m_1} + b_2\phi^{m_2}\big(\phi^2(2R - 4{R^a}_a + 2{R^{ab}}_{ab}) + 2\Phi - 4{\Phi^a}_a + 2{\Phi^{ab}}_{ab}\big)\Big]$$

$$\frac{\partial^2 \mathcal{L}_m}{\partial S_{zizj}\partial S_{\bar{z}k\bar{z}l}} K_{zij}K_{\bar{z}kl} = 2b_2\phi^{m_2}\Big[2K_aK^a - 2K_{aij}K^{aij}\Big],$$

Result

Inserting these results in the Camps-Dong formula, and defining $a(s)$ and $b(s)$ as certain numerical factors, we can write:

$$S_A = \frac{Area}{4G_N} + S_m,$$

where:

$$S_m = 4\pi \int d^3y\sqrt{h}\,\Big[b_1\phi^{m_1+2} + b_2\big(\mathcal{R}\phi^{m_2+4} + a(s)\ \phi^{m_2+3}h^{ij}\partial_i\partial_j\phi \\ + b(s)\ \phi^{m_2+2}h^{ij}\ \partial_i\phi\partial_j\phi\phi - \frac{12}{s^2}\ \phi^{m_2+2}\nabla_z\phi\nabla_{\bar{z}}\phi\big)\Big].$$

Having the explicit form of the coefficients, we recognize the second, third and fourth terms as the terms appearing in the intrinsic S-tensor, defined on the codim-2 surface, so:

$$\boxed{S_m = 4\pi \int dy^{D-2}\sqrt{h}\big(b_1\phi^{m_1+2} + b_2\phi^{m_2+2}\ (\mathcal{S} - \frac{12}{s^2}\nabla_z\phi\nabla_{\bar{z}}\phi)\big)}$$

Calculation by the squashed cone method

Alternatively, we can use explicitly the squashed cone method. First expand the Lagrangian in terms of $R_{\mu\nu}^{\rho\lambda}$ and ϕ, and look for those terms in the action which are proportional to ϵ after integration respect to the transverse coordinates:

$$I_R \propto \beta_1 \int_{\mathcal{M}} d^5x\sqrt{-g}\ \phi^{15} R, \tag{22}$$

$$I_H \propto \beta_2 \int_{\mathcal{M}} d^5x\sqrt{-g}\ (R_{ab} + \frac{1}{8} g_{ab} R)\ \phi^3 \nabla^a\phi\nabla^b\phi, \tag{23}$$

$$I_{H'} \propto \beta_2 \int_{\mathcal{M}} d^5x\sqrt{-g}\ G_{ab}\ \phi^4 \nabla^a\nabla^b\phi, \tag{24}$$

$$I_{GB} \propto \beta_2 \int_{\mathcal{M}} d^5x\sqrt{-g}\ \phi^5 \left(R_{cdab}R^{cdab} - 4R_{ab}R^{ab} + R^2\right), \tag{25}$$

$$I_M = \beta_2 \int_{\mathcal{M}} d^5x\sqrt{-g}\ \left(\frac{36}{s^2}\phi\nabla_a\nabla_b\phi\nabla^a\nabla^b\phi - \frac{12}{s^2}(\phi\Box\phi)^2\right) \tag{26}$$

Calculation by the squashed cone method

The result using this method is:

$$\boxed{S_m = 4\pi \int dy^{D-2}\sqrt{h}\Big(b_1\phi^{m_1+2} + b_2\phi^{m_2+2}\ (\mathcal{S} + \frac{24}{s^2}\nabla_z\phi\nabla_{\bar{z}}\phi)\Big)}$$

Note that the coefficients of the last term, obtained by the Camps-Dong formula, and by the squashed cone method are not the same.

What is wrong?

The reason is that the Camps-Dong formula does not consider possible anomalous terms coming from the terms in the Lagrangian which *only* contain the scalar field an its derivatives.

Anomalous contribution from the scalar field

In fact, the term in the action which contributes to S_m and only depends on derivatives of the scalar field has a quadratic dependence on ϵ which is promoted to a linear one, exactly as in the gravity case which gives rise to the term

$$2\pi \int d^{D-2}y\sqrt{h}\sum_{\alpha}\left(\frac{\partial^2 \mathcal{L}}{\partial R_{zizj}\partial R_{\bar{z}k\bar{z}l}}\right)_{\alpha}\frac{K_{zij}K_{\bar{z}kl}}{q_\alpha + 1}$$

For this particular case, we can match the two results if we add the following term to the Camps-Dong formula:

$$S_{AnomalyMatter} = 2\pi \int d^{D-2}y\sqrt{h}\left(\frac{\partial^2 \mathcal{L}}{\partial(\nabla_z\nabla_z\phi)\ \partial(\nabla_{\bar{z}}\nabla_{\bar{z}}\phi)}\nabla_z\phi\nabla_{\bar{z}}\phi\right) \tag{27}$$

We conjecture that this is the correction to the Camps-Dong formula corresponding to a general gravity theory depending only on the Riemann tensor, and non-minimally coupled with a scalar field.

Future work

- It remains to be determined if $S_{AnomalyMatter}$ is the only anomalous contribution that appears when we couple a scalar field to a gravity theory, or in any other case, to find those other contributions.
- The anomalous term consists of an infinite expansion at $n > 1$.This is dubbed as the *splitting problem*. To write this contribution explicitly in terms of the field and its derivatives at $n = 1$, we have to solve the most divergent part of the equations of motion.
- Finally we would like to probe that the codimension-2 surface in which this functional has to be evaluated is obtained by minimization of the same. For this, we have to extract the divergences in the equations of motion for the theory. Having the explicit form of the HEE functional, we can evaluate the entanglement entropy for the quantum dual of the 5-dimensional black hole reviewed previously.