# Causality, T-duality and high-derivative gravity theories

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X. Camanho, J. Edelstein, J. Maldacena, A. Zhiboedov, *Journal of High Energy Physics* 1602 (2016) 020

J. Edelstein, G. Giribet, C. Gómez, E. Kilicarslan, M. Leoni, B. Tekin, *Physical Review D* 95 (2017) 104016

J. Edelstein, K. Sfetsos, J. A. Sierra-García, A. Vilar López, *Journal of High Energy Physics* 1806 (2018) 142

## Constraints on low-energy effective theories

Not all local Lorentz invariant Lagrangians are consistent. For instance, if we consider a $U(1)$ gauge field, Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi, 2006

$$\mathcal{I} = \int d^4x\sqrt{-\mathrm{G}}\,\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{c_1}{\Lambda_\star^4}(F_{\mu\nu}F^{\mu\nu})^2 + \frac{c_2}{\Lambda_\star^4}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2 + \cdots\right],$$

the coefficients $c_1$ and $c_2$, a priori unconstrained, give the leading S-matrix amplitudes at energies beneath $\Lambda_\star$.

To avoid analiticity and unitarity problems of the S-matrix, and superluminal modes, they must be possitive.

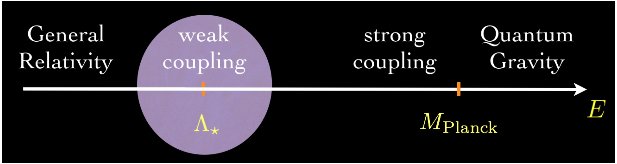
They are indeed positive if obtained by integrating out electrons in QED.

In the case of gravity, the situation is tighter. If we have

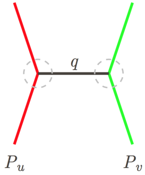
$$\mathcal{I} = \int d^Dx\sqrt{-\mathrm{G}}\,\left[R + \frac{c_1}{\Lambda_\star^2}R_{\mu\nu\sigma\lambda}R^{\mu\nu\sigma\lambda} + \frac{c_2}{\Lambda_\star^4}R_{\mu\nu\sigma\lambda}R^{\sigma\lambda\alpha\beta}R_{\alpha\beta}{}^{\mu\nu} + \cdots\right],$$

then $c_1, c_2 \simeq 0$ or, else, an infinite tower of massive higher-spin particles must be introduced. Camanho, Edelstein, Maldacena, Zhiboedov, 2016

The setup we are dealing with is represented as follows:




The idea is to compute the tree-level 4-point amplitude




in the Eikonal limit: $s \simeq P_u P_v$ very large (yet $s \ll M^2_{\text{Planck}}$), $t \simeq -\vec{q}^{\,2} \ll s$, and fixed impact parameter $\vec{b}$.

## Constraints on low-energy effective theories

The eikonal phase shift, in the impact parameter representation, is given by

$$\begin{aligned}\delta(s,\vec{b}) &= \frac{1}{2s}\int \frac{d^{D-2}\vec{q}}{(2\pi)^{D-2}}\, e^{i\vec{b}\cdot\vec{q}}\mathcal{A}^{[4]}_{\rm tree}(s,t) \\ &= \frac{1}{2s}\mathcal{A}^{[3]}_{13I}(-i\partial_{\vec{b}})\mathcal{A}^{[3]}_{I24}(-i\partial_{\vec{b}})\int \frac{d^{D-2}\vec{q}}{(2\pi)^{D-2}}\,\frac{e^{i\vec{b}\cdot\vec{q}}}{\vec{q}^2} \\ &= \frac{\Gamma\left(\frac{D-4}{2}\right)}{4\pi^{\frac{D-2}{2}}}\,\frac{1}{2s}\mathcal{A}^{[3]}_{13I}(-i\partial_{\vec{b}})\mathcal{A}^{[3]}_{I24}(-i\partial_{\vec{b}})\frac{1}{|\vec{b}|^{D-4}}\ ,\end{aligned}$$

where the vertices $\mathcal{A}^{[3]}_{13I} := \mathcal{A}^{[3]}_{13I}(-i\partial_{\vec{b}})$ and $\mathcal{A}^{[3]}_{I24} := \mathcal{A}^{[3]}_{I24}(-i\partial_{\vec{b}})$ read:

$$\begin{aligned}\mathcal{A}^{[3]}_{13I} &= 2P_u^2\left[(\vec{e}_1\cdot\vec{e}_3)^2 + \frac{c_1}{\Lambda_\star^2}(\vec{e}_1\cdot\vec{e}_3)(\vec{e}_1\cdot\partial_{\vec{b}})(\vec{e}_3\cdot\partial_{\vec{b}}) + \frac{c_2}{\Lambda_\star^4}(\vec{e}_1\cdot\partial_{\vec{b}})^2(\vec{e}_3\cdot\partial_{\vec{b}})^2\right], \\ \mathcal{A}^{[3]}_{I24} &= 2P_v^2\left[(\vec{e}_2\cdot\vec{e}_4)^2 + \frac{c_1}{\Lambda_\star^2}(\vec{e}_2\cdot\vec{e}_4)(\vec{e}_2\cdot\partial_{\vec{b}})(\vec{e}_4\cdot\partial_{\vec{b}}) + \frac{c_2}{\Lambda_\star^4}(\vec{e}_2\cdot\partial_{\vec{b}})^2(\vec{e}_4\cdot\partial_{\vec{b}})^2\right].\end{aligned}$$

## Constraints on low-energy effective theories

Then, the eikonal phase shift scales linearly, $\delta(s,\vec{b}) \sim s$. The S-matrix reads

$$S(s,\vec{b}) \simeq e^{i\delta(s,\vec{b})} \ ,$$

and it has to be analytic and bounded,

$$|S(s,\vec{b})| \leq 1 \ , \qquad \text{for} \quad \mathbb{I}\mathrm{m}\, s \geq 0 \ .$$

It is impossible to fulfill this condition for all polarizations of the intervening on-shell gravitons. Camanho, Edelstein, Maldacena, Zhiboedov, 2016

In fact, $\delta(s,\vec{b})$ is proportional to the Shapiro delay, which relates this problem to causality violation.

New degrees of freedom at $E \sim \Lambda_\star$; Spin $J$ particles contribute $\delta(s,\vec{b}) \sim s^{J-1}$.

Massive $J = 2$ would do the job but do not conserve angular momentum!

We need an infinite tower of higher-spin particles with a delicate fine-tuning in their couplings, as in string theory! D'Appolonio, Di Vecchia, Russo, Veneziano, 2015

## The three-dimensional case

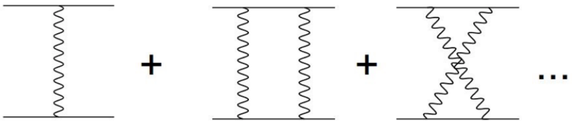
General Relativity in 3D has no local degrees of freedom. Spacetime is locally flat outside the locus of any source; there should be no Shapiro delay.

A shock wave solution reads Edelstein, Giribet, Gómez, Kilicarslan, Leoni, Tekin, 2017

$$ds^2 = -du dv + 2\sigma |p| \delta(u) \theta(-y) y du^2 + dy^2 \ ,$$

where $\sigma = 8\pi G$ is just a sign, $\sigma^2 = 1$: the Shapiro delay vanishes, $\Delta v = 0$.

We can equivalently compute the Eikonal scattering amplitude of a scalar field minimally coupled to gravity,




which leads to

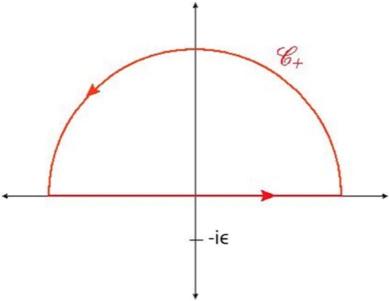
$$\delta(b,s) = \frac{1}{2s} \int \frac{dq}{2\pi} e^{iqb} \mathcal{A}_{\rm tree}(s,-q^2) \ , \qquad \text{where} \quad \mathcal{A}_{\rm tree}(s,-q^2) = -\sigma \frac{s^2}{t} \ .$$

## The three-dimensional case

The Eikonal phase shift naively diverges,

$$\delta(b,s) = \frac{\sigma s}{2}\int \frac{dq}{2\pi}\frac{e^{iqb}}{q^2} \ ,$$

unless we conveniently regularize this expression as




$$\delta(b,s) = \lim_{\epsilon\to 0}\frac{\sigma s}{2}\oint_{\mathcal{C}_+} = \lim_{\epsilon\to 0}\frac{\sigma s}{2}\int \frac{dq}{2\pi}\frac{e^{iqb}}{(q+i\epsilon)^2} = 0 \ .$$

## The three-dimensional case: Topologically Masive Gravity

The action of Topologically Massive Gravity reads Deser, Jackiw, Templeton, 1982

$$\mathcal{I}_{\rm TMG} = \int d^3x \left[\sigma\sqrt{-g}R + \frac{1}{2\mu}\epsilon^{\mu\nu\rho}\Gamma^{\sigma}_{\mu\alpha}\left(\partial_\nu\Gamma^{\alpha}_{\rho\sigma} + \frac{2}{3}\Gamma^{\alpha}_{\nu\lambda}\Gamma^{\lambda}_{\rho\sigma}\right) + \frac{1}{2}\sqrt{-g}\,\partial_\mu\phi\partial^\mu\phi\right],$$

leading to the equations of motion:

$$\sigma G_{\mu\nu} + \frac{1}{\mu}\frac{\epsilon_\mu{}^{\alpha\beta}}{\sqrt{-g}}\nabla_\alpha\left(R_{\beta\nu} - \frac{1}{4}g_{\beta\nu}R\right) = T_{\mu\nu}\ .$$

TMG violates parity: it has a single massive spin-2 graviton mode with

$$m_g = \mu > 0\ .$$

whose kinetic term is well defined for $\sigma = -1$.

The shock wave solution can be found to be

$$h(u,y) = -\frac{2\sigma|p|}{\mu}\delta(u)\theta(y)e^{-\mu y} + \frac{2\sigma|p|}{\mu}\delta(u)\theta(-y)\left(\mu y - 1\right)\ .$$

## The three-dimensional case: Topologically Masive Gravity

A scalar particle crossing the shock wave experiences

$$\Delta v = -\frac{2\sigma|p|}{\mu} e^{-\mu b} > 0 \ .$$

while the massive graviton suffers a similar Shapiro delay,

$$\Delta v = -\frac{3\sigma|p|}{m_g} e^{-m_g b} > 0 \ .$$

Both unitarity and causality demand negativity of the Newton constant.

A comment is in order: we need to fix the gauge to deal with asymptotically flat and Cartesian coordinates, which we can do for $y > 0$ or $y < 0$.

We can again compute the Eikonal scattering amplitude from

$$\mathcal{I}_{\rm TMG} = \int d^3x \left[\sigma\sqrt{-g}R + \frac{1}{2\mu}\epsilon^{\mu\nu\rho}\Gamma^{\sigma}_{\mu\alpha}\left(\partial_\nu\Gamma^{\alpha}_{\rho\sigma} + \frac{2}{3}\Gamma^{\alpha}_{\nu\lambda}\Gamma^{\lambda}_{\rho\sigma}\right) + \frac{1}{2}\sqrt{-g}\partial_\mu\phi\partial^\mu\phi\right] ,$$

whose vertex reads

$$V(p,p') = \frac{1}{2}\left(p_\mu p'_\nu + p_\nu p'_\mu - \eta_{\mu\nu}(p\cdot p' + \mu^2)\right) \ .$$

## The three-dimensional case: Topologically Masive Gravity

The graviton propagator in de Donder gauge

$$
\begin{aligned}
\mathcal{D}_{\mu\nu\alpha\beta} \;=\; & \frac{4i\mu(-p^2)}{(-p^2)^3-\mu^2(-p^2)^2}\Big[-\frac{\sigma\mu}{2}\left(\Theta_{\mu\alpha}\Theta_{\nu\beta}+\Theta_{\mu\beta}\Theta_{\nu\alpha}-\Theta_{\mu\nu}\Theta_{\alpha\beta}\right) \\
& -\frac{1}{4}\left(\epsilon_{\mu\alpha\lambda}\Theta_{\beta\nu}+\epsilon_{\mu\beta\lambda}\Theta_{\alpha\nu}+\epsilon_{\nu\alpha\lambda}\Theta_{\beta\mu}+\epsilon_{\nu\beta\lambda}\Theta_{\alpha\mu}\right)(ip^\lambda)\Big] \\
& +\frac{4i\sigma}{(-p^2)}\Big[\frac{1}{2}\left(\Theta_{\mu\alpha}\omega_{\nu\beta}+\Theta_{\mu\beta}\omega_{\nu\alpha}+\Theta_{\nu\alpha}\omega_{\mu\beta}+\Theta_{\nu\beta}\omega_{\mu\alpha}\right) \\
& -\frac{1}{2}\Theta_{\mu\nu}\Theta_{\alpha\beta}-\left(\Theta_{\mu\nu}\omega_{\alpha\beta}+\Theta_{\alpha\beta}\omega_{\mu\nu}\right)\Big]
\end{aligned}
$$

where

$$
\omega=\partial_\mu\partial_\nu\,, \qquad \text{and} \qquad \Theta_{\mu\nu}=\eta_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{\Box}\,.
$$

The tree-level amplitude reads

$$
\mathcal{A}_{\text{tree}}(s,t)=-\sigma\frac{s^2}{t}\frac{1}{1-\frac{i\sigma}{\mu}\sqrt{-t}}\,.
$$

## The three-dimensional case: Topologically Masive Gravity

The Eikonal phase shift naively diverges but can be regularized

$$\begin{aligned}\delta(b,s) &= \frac{1}{2s}\int \frac{dq}{2\pi} e^{iqb}\mathcal{A}_{\text{tree}}(s,t=-q^2)\\ &= \lim_{\epsilon\to 0}\frac{i\mu s}{2}\int\frac{dq}{2\pi}\frac{e^{i(q+i\epsilon)b}}{(q+i\epsilon)^2(q+i\epsilon+i\sigma\mu)}\end{aligned}$$

So that

$$\delta(b,s)\sim\Delta v=-\frac{2\sigma|p|}{\mu}e^{-\mu b}>0\,.$$

What happens in the AdS case?

In principle, similar results hold, although the sign of the Newton constant enters into conflict with that needed for black hole thermodynamics.

Let us scrutinize whether T-duality can be used as a restrictive tool to classify well-behaved theories, and whether it resolves the *sign problem*.

## T-duality and high-derivative three-dimensional gravity

In String Theory, T-duality is a symmetry of the low energy effective action to all orders in $\alpha'$. Consider the action of the massless NS-NS sector,

$$\mathcal{I}_0 = \int d^D x \sqrt{-\mathrm{G}}\, e^{-2\Phi} \mathcal{L}^{(0)} \ ,$$

(we have set $16\pi G_{\mathrm{N}} = 1$), where

$$\mathcal{L}^{(0)} = R - 2\Lambda + 4(\nabla_M \nabla^M \Phi - \nabla_M \Phi \nabla^M \Phi) - \frac{1}{12} H_{MNR}\, H^{MNR} \ .$$

If our fields possess a $\mathrm{U}(1)$ isometry along, say, $\psi$, the set of transformations

$$\widehat{\mathrm{G}}_{\psi\psi} = 1/\mathrm{G}_{\psi\psi} \ , \quad \widehat{\mathrm{G}}_{\psi\mu} = B_{\psi\mu}/\mathrm{G}_{\psi\psi} \ , \quad \widehat{B}_{\psi\mu} = \mathrm{G}_{\psi\mu}/\mathrm{G}_{\psi\psi} \ ,$$

$$\widehat{\mathrm{G}}_{\mu\nu} = \mathrm{G}_{\mu\nu} - (\mathrm{G}_{\psi\mu}\mathrm{G}_{\psi\nu} - B_{\psi\mu}B_{\psi\nu})/\mathrm{G}_{\psi\psi} \ ,$$

$$\widehat{B}_{\mu\nu} = B_{\mu\nu} - (\mathrm{G}_{\psi\mu}B_{\psi\nu} - \mathrm{G}_{\psi\nu}B_{\psi\mu})/\mathrm{G}_{\psi\psi} \ , \quad \widehat{\Phi} = \Phi - \frac{1}{2}\log \mathrm{G}_{\psi\psi} \ ,$$

leave $\mathcal{I}_0$ invariant; hatted fields are those of the T-dual background. Buscher, 1987

# T-duality and high-derivative three-dimensional gravity

A more transparent expression emerges from explicitly writing of the fields in terms of those dimensionally reduced along $\psi$,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + e^{2\sigma}(d\psi + V_\mu\, dx^\mu)^2 \ ,$$

$$B_{\mu\nu} = \mathcal{B}_{\mu\nu} + \frac{1}{2}\left(W_\mu V_\nu - W_\nu V_\mu\right) , \qquad B_{\mu\psi} = W_\mu \ ,$$

$$\Phi = \phi + \frac{1}{2}\sigma \ ,$$

where $V_\mu$ and $W_\mu$ are the $U(1)$ gauge vectors arising from the metric and the $B$-field with an index along the cyclic coordinate. Kaloper, Meissner, 1997

$\mathcal{B}_{\mu\nu}$ is the dimensionally reduced Kalb-Ramond field, and the $W_{[\mu} V_{\nu]}$-term is a convenient field redefinition; now, Maharana, Schwarz, 1993

$$V_\mu \leftrightarrow W_\mu \ , \qquad \text{and} \qquad \sigma \leftrightarrow -\sigma \ ,$$

are the T-duality rules; they square to the identity.

Does T-duality provide a sufficient constraint to obtain the next-to-leading term in a low-energy expansion? Edelstein, Sfetsos, Sierra-García, Vilar López, 2018

There is a two parameter Lagrangian based on the $O(D, D)$ invariant Double Field Theory realization of T-duality, Hohm, Zwiebach, 2014 Marqués, Núñez, 2015

$$\mathcal{L}^{(1)} = \gamma_+ \mathcal{L}_+^{(1)} + \gamma_- \mathcal{L}_-^{(1)} ,$$

where $\gamma_\pm$ are assumed to be order $\Lambda_\star^{-2}$ parameters, and

$$\begin{aligned} \mathcal{L}_+^{(1)} &= \frac{1}{2} R_{MNRS} R^{MNRS} - \frac{3}{4} H^{MNR} H_{MSL} R_{NR}{}^{SL} + \frac{1}{6} \nabla_M H_{NRS} \nabla^M H^{NRS} \\ &\quad + \frac{1}{48} H^{MNR} H_{MS}{}^L H_{NL}{}^T H_{RT}{}^S + \frac{1}{16} H_{MRT} H^{MR}{}_L H_{NS}{}^T H^{NSL} \\ &\quad - \frac{1}{2} H^{MNR} \partial_M \left( H_{NAB} \Omega_R{}^{AB} \right) , \\ \mathcal{L}_-^{(1)} &= H^{MNR} \Theta_{MNR} . \end{aligned}$$

$\mathcal{L}_+^{(1)}$ was constructed (modulo field redefinitions) long ago. Metsaev, Tseytlin, 1987

# T-duality and high-derivative three-dimensional gravity

The sign subindex in $\mathcal{L}_{\pm}^{(1)}$ makes reference to its parity under $H \to -H$.

$\Theta_{MNR}$ is the gravitational Chern-Simons three-form,

$$\Theta_{MNR} = \Omega_{[MA}{}^{B}\,\partial_N\,\Omega_{R]B}{}^{A} + \frac{2}{3}\Omega_{[MA}{}^{B}\,\Omega_{NB}{}^{C}\,\Omega_{R]C}{}^{A}\ ,$$

$\Omega_{MA}{}^{B}$ being the Lorentz connection. Some particular values of $\gamma_{\pm}$ correspond to low-energy effective actions coming from string theories:

$$\begin{aligned} \gamma_- &= 0 && \text{bosonic}\ , \\ \gamma_+ &= -\gamma_- && \text{heterotic}\ , \\ \gamma_+ = \gamma_- &= 0 && \text{type II}\ . \end{aligned}$$

In the former two cases, $\gamma_+ - \gamma_- = \frac{1}{2}\alpha'$.

The case $\gamma_+ = \gamma_-$ is also special. Hohm, Siegel, Zwiebach, 2014.

For other $\gamma_+$ and $\gamma_-$, the Lagrangian is not known to be related to a $\sigma$-model or a CFT. Yet, it is invariant under T-duality if we neglect $\mathcal{O}(\gamma_{\pm}^2)$ terms.

## T-duality and high-derivative three-dimensional gravity

I want to explore the behavior of the $\gamma_\pm$-corrected BTZ black hole solving

$$\mathcal{I} = \int d^3x \sqrt{-\mathrm{G}}\, e^{-2\Phi} \left[ \mathcal{L}^{(0)} + \gamma_+ \mathcal{L}_+^{(1)} + \gamma_- \mathcal{L}_-^{(1)} \right] ,$$

when it is subjected to a T-duality transformation.

The vanishing $\gamma_\pm$ was studied by Horowitz and Welch. Horowitz, Welch, 1994

They found that the BTZ black hole is mapped onto a black string. Despite the geometry being severely modified, the existence of a bifurcate Killing horizon, the temperature and the entropy of both solutions are the same.

I will show that the temperature and entropy of the BTZ black hole coincide with those of the T-dual black string when $\gamma_\pm$ corrections are included.

The value of these parameters not being restricted to the string theory values!

Although T-duality does not provide stringent enough constraints on these parameters, AdS/CFT does.

# The BTZ black hole and $\gamma_\pm$ corrections

To discuss the effect of $\gamma_\pm$ corrections on the BTZ black hole, we write down the equations of motion starting from an equivalent action: Bergshoeff, deRoo, 1989

$$\mathcal{I} = \int d^3x\sqrt{-\mathrm{G}}\, e^{-2\Phi}\Big[R + \frac{4}{\ell^2} + 4(\nabla_M\nabla^M\Phi - \nabla_M\Phi\nabla^M\Phi)$$
$$-\frac{1}{12}\bar{H}_{MNR}\,\bar{H}^{MNR} + \frac{1}{8}\sum_{k=\pm} a_k\, R^{(k)}_{MNA}{}^{B} R^{(k)MN}{}_{B}{}^{A}\Big] ,$$

where the sum runs over two signs, $k = \pm$, and the parameters $a_\pm$ are going to be dubbed $a_- \equiv a$ and $a_+ \equiv b$.

$$\bar{H}_{MNR} := H_{MNR} - \frac{3}{2}\left(a\,\Theta^{(-)}_{MNR} - b\,\Theta^{(+)}_{MNR}\right) ,$$

$\Theta^{(\pm)}$ denote the gravitational Chern-Simons form of the Lorentz connection with torsion $\Omega^{(\pm)}_{MA}{}^{B} = \Omega_{MA}{}^{B} \pm \frac{1}{2}H_{MA}{}^{B}$.

The parameters $\gamma_\pm$ are related to $a$ and $b$ in a simple fashion:

$$\gamma_\pm = \mp\frac{a \pm b}{4} .$$

# The BTZ black hole and $\gamma_\pm$ corrections

From the variation of $\mathcal{I}$ one gets the equations of motion:

$$R + \frac{4}{\ell^2} + 4(\nabla^2\Phi - (\nabla\Phi)^2) - \frac{1}{12}\bar{H}_{MNR}\bar{H}^{MNR} + \frac{1}{8}\sum_{k=\pm} a_k R^{(k)}_{MNA}{}^{B} R^{(k)MN}{}_{B}{}^{A} = 0\,,$$

$$\nabla_M\left[e^{-2\Phi}\bar{H}^{MNR} + \frac{3}{2}\sum_{k=\pm} a_k\left(e^{-2\Phi}H^{ST[M}R^{(k)RN]}_{ST} - k\nabla^{(k)}_S\left[e^{-2\Phi}R^{(k)S[MNR]}\right]\right)\right] = 0\,,$$

$$R_{MN} + 2\nabla_M\nabla_N\Phi - \frac{1}{4}\bar{H}_{MRS}\bar{H}_N^{RS} - \frac{1}{4}\sum_{k=\pm} a_k\Big[R^{(k)}_{MRST}R^{(k)RST}_N$$

$$+e^{2\Phi}\left(2\mathrm{G}_{S(M|}\nabla_R + kH_{RS(M|}\right)\left(\delta_U{}^S\nabla^{(k)}_T + kH_{TU}{}^S\right)\left(e^{-2\Phi}R^{(k)TUR}{}_{|N)}\right)\Big] = 0\,,$$

where $\nabla^2 = \nabla_M\nabla^M$, $(\nabla\Phi)^2 = \nabla_M\Phi\nabla^M\Phi$, and $\nabla^{(k)}$ is the covariant derivative involving the connection with torsion $\Omega^{(k)}_{MA}{}^{B}$; *i.e.*, $\nabla^{(\pm)}$ is the (diffeomorphism) covariant derivative with torsionful Christoffel symbols $\Gamma^{(\pm)R}_{MN} = \Gamma^R_{MN} \mp \frac{1}{2}H_{MN}{}^R$.

We want to find exact black hole solutions of this system. The BTZ black hole can be *dressed* to give a solution of the equations arising from $\mathcal{I}_0$: Horowitz, Welch, 1993

## The BTZ black hole and $\gamma_\pm$ corrections

$$ds^2 = -N^2 dt^2 + \frac{dr^2}{N^2} + r^2 \left(d\psi + N^\psi dt\right)^2 ,$$

$$e^{-2\Phi} = 1 , \qquad B_{t\psi} = \frac{r_+^2 - r^2}{\ell} ,$$

while all other components vanish; the lapse is the standard BTZ

$$N^2 = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{\ell^2 r^2} , \qquad N^\psi = \frac{r_+ r_-}{\ell} \left(\frac{1}{r_+^2} - \frac{1}{r^2}\right) ,$$

where $r_+$ and $r_-$ are two integration constants, better expressed in terms of the ($\gamma_\pm$-uncorrected) black hole mass and angular momentum,

$$M = \frac{r_+^2 + r_-^2}{\ell^2} , \qquad J = \frac{2 r_+ r_-}{\ell} ,$$

which agree with the standard BTZ expressions. For the stringy values, this solution is exact to all orders in $\alpha'$. Horne,Horowitz, Steif, 1992

This is a solution for arbitrary $\gamma_\pm$, because $\Theta^{(\pm)}_{MNR}$ and $R^{(\pm)}_{MNAB}$ vanish when evaluated on the BTZ, as expected.

## The BTZ black hole and $\gamma_\pm$ corrections

In spite of the fact that the action can be shown to be local Lorentz invariant on-shell, it is not so for an arbitrary background.

Indeed, it is invariant under an anomalous local Lorentz transformation:

$$\delta_\Lambda E_M{}^A = E_M{}^B \Lambda_B{}^A \, , \qquad \delta_\Lambda B_{MN} = -\frac{a}{2}\, \partial_{[M}\Lambda_A{}^B \Omega^{(-)\,A}_{N]B} + \frac{b}{2}\, \partial_{[M}\Lambda_A{}^B \Omega^{(+)\,A}_{N]B} \, ,$$

where $\Lambda_A{}^B$ is the infinitesimal generator.

For the heterotic string case such transformation is due to the Green-Schwarz anomaly cancellation mechanism.

It leaves $\bar{H}_{MNR}$ invariant; the rest of the action is manifestly invariant.

Due to this anomalous Lorentz invariance, it is necessary to specify both the vierbein and the $B$-field which solve the equations of motion.

A different frame has to be accompanied by a compensating transformation in the $B$-field so that $\bar{H}_{MNR}$ remains invariant.

## The BTZ black hole and $\gamma_\pm$ corrections

Consistency of the black hole thermodynamics requires the horizon to be a regular bifurcate Killing horizon. Racz, Wald, 1992

Consider a stationary solution with a spacelike isometry, $\eta$, whose orbits are closed and provide the symmetry along which we apply T-duality.

Furthermore, $\eta$ commutes with an asymptotically timelike Killing field $\lambda$,

$$\lambda = \partial_t \,, \qquad \eta = \partial_{\psi'} \,.$$

For some constant $\omega$ there is a compact spacelike codimension-2 bifurcate Killing horizon $\mathcal{B}$ in which

$$\xi\big|_{\mathcal{B}} = \lambda + \omega\, \eta\big|_{\mathcal{B}} = 0 \,,$$

$\xi$ being the null Killing vector that generates the Killing horizon.

All fields must obey regularity conditions at $\mathcal{B}$. Wald, 1993 Jacobson, Kang, Myers, 1994

The particular choice of coordinates, as well as the gauge choice for the $B$-field, must be such that $\mathrm{G}_{t\psi}\big|_{\mathcal{N}} = B_{t\psi}\big|_{\mathcal{N}} = 0$ at the horizon $\mathcal{N}$.

# The BTZ black hole and $\gamma_\pm$ corrections

Otherwise, either the original solution or its T-dual along $\psi$ are singular at leading order in $\gamma_\pm$. Horowitz, Welch, 1994

Given the fact that $\Omega_{MA}{}^{B}$ enters explicitly in the action, we further impose:

$$\xi^M \Omega_{MA}{}^{B}\big|_{\mathcal{N}} = \Omega_{tA}{}^{B}\big|_{\mathcal{N}} = 0 \ .$$

This follows from $\xi\big|_{\mathcal{B}} = \partial_t\big|_{\mathcal{B}} = 0$, and the regularity of $\Omega_A{}^B$ as 1-forms.

It turns out that the obvious vierbein choice of our solution,

$$e^0 = N dt \ , \qquad e^1 = \frac{dr}{N} \ , \qquad e^2 = r(d\psi + N^\psi dt) \ ,$$

does not fulfill this condition, $\Omega_{t0}{}^{1}\big|_{\mathcal{N}} = -\kappa \neq 0$.

$\Omega_{MA}{}^{B}$ can be made regular, $\Omega_{tA}{}^{B}\big|_{\mathcal{N}} = 0$, by a Lorentz transformation:

$$E^0 = \cosh(\kappa t)\, e^0 + \sinh(\kappa t)\, e^1 \ , \qquad E^1 = \sinh(\kappa t)\, e^0 + \cosh(\kappa t)\, e^1 \ .$$

Also, $B_{t\psi} \to B_{t\psi} + \Delta B_{t\psi}$.

## The $\gamma_\pm$-corrected T-dual black strings

It is possible to perform a non-covariant field redefinition

$$\tilde{\mathrm{G}}_{MN} = \mathrm{G}_{MN} - \frac{1}{4}\sum_{k=\pm} a_k\, \Omega^{(k)\ B}_{MA}\, \Omega^{(k)\ A}_{NB} \,,$$

$$\tilde{B}_{MN} = B_{MN} \,, \qquad e^{-2\tilde{\Phi}}\sqrt{-\tilde{\mathrm{G}}} = e^{-2\Phi}\sqrt{-\mathrm{G}} \,.$$

that keeps the Buscher rules unaffected Marqués, Núñez, 2015.

In this fashion, a $\gamma_\pm$ dependence is inherited by the T-dual solution:

$$ds^2 = -N^2 dt^2 + \frac{dr^2}{N^2} + e^{-2\sigma}\,(d\chi + N^\chi\, dt)^2 \,, \qquad e^{-2\Phi} = r^2(1+\Delta_+) \,,$$

$$B_{t\chi} = \frac{e}{\ell}\left(1 - \frac{r_+^2}{r^2}\right) - \frac{2}{\ell r^2}\left(\gamma_+ \frac{J}{\ell}\left(1 - \frac{M\ell^2}{2r^2}\right) + \gamma_-\left(M - \frac{J^2}{2r^2}\right)\right) \,,$$

where $N^2$ is the same lapse function, $e = \frac{r_-}{r_+}$, and

$$N^\chi = \frac{r_+^2 - r^2}{\ell}\,(1 + e\Delta_-) \,, \qquad \Delta_\pm = \frac{2}{r^2}(\gamma_\pm M + \gamma_\mp J/\ell) \,, \qquad e^{-\sigma} = \frac{1-\Delta_+}{r} \,.$$

This solution represents a $\gamma_\pm$-corrected black string.

## The $\gamma_\pm$-corrected T-dual black strings

The $\gamma_\pm = 0$ case corresponds to HW solution. Horowitz, Welch, 1994

The bifurcated Killing horizon is not regular, though.

Regularity requires either applying T-duality to the regular BTZ black hole or, equivalently, by performing a local Lorentz boost.

It has to be accompanied by $B_{t\chi} \to B_{t\chi} + \Delta B_{t\chi}$,

$$\Delta B_{t\chi} = \frac{\kappa}{\ell r^2}\left(\frac{\ell J}{r}\gamma_+ + 2r\gamma_-\right).$$

To better connect our result with those in the literature,

$$\begin{aligned} t &= \ell(r_+^2 - r_-^2)^{-1/2}\left(1 - 2\ell^{-2}\left(\gamma_+ - e\gamma_-\right)\right)(T + X)\,, \\ \chi &= -(r_+^2 - r_-^2)^{1/2}\left[\left(1 + 2\ell^{-2}\left(\gamma_+ + e\gamma_-\right)\right)X + 4\ell^{-2}e\gamma_-\,T\right], \end{aligned}$$

and let us change the radial variable: $r^2 = \ell\rho$.

We obtain the $\gamma_\pm$-corrected T-dual black string:

## The $\gamma_\pm$-corrected T-dual black strings

$$
\begin{aligned}
ds^2 &= -\left(1-\frac{\mathcal{M}}{\rho}\right)\left[1-\frac{4\mathcal{M}}{\ell^2\rho}\left(\gamma_+ + e\gamma_-\right)\right]dT^2 - \frac{4\left(1-e^2\right)\gamma_-\mathcal{Q}}{\ell^2\rho}\left(1+\frac{\mathcal{M}}{\rho}\right)dT\,dX \\
&\quad + \left(1+\frac{e\mathcal{Q}}{\rho}\right)\left[1+\frac{4\mathcal{Q}}{\ell^2\rho}\left(\gamma_- + e\gamma_+\right)\right]dX^2 + \left(1-\frac{\mathcal{M}}{\rho}\right)^{-1}\left(1+\frac{e\mathcal{Q}}{\rho}\right)^{-1}\frac{\ell^2 d\rho^2}{4\rho^2}\,, \\
e^{-2\Phi} &= \ell\rho + 2\ell^{-1}\left[\left(1+e^2\right)\gamma_+\mathcal{M} - 2\gamma_-\mathcal{Q}\right], \\
B_{TX} &= -e - \frac{\mathcal{Q}}{\rho} + \frac{4\mathcal{Q}}{\ell^2\rho}\left(\gamma_+\left(\frac{\mathcal{M}-e\mathcal{Q}}{2\rho}-1\right) - \gamma_-\left(\frac{e+e^{-1}}{2}+\frac{\mathcal{Q}}{\rho}\right)\right), \\
\Delta B_{TX} &= 2\kappa\ell^{-1/2}\left(\gamma_+\mathcal{Q}\rho^{-3/2} - \gamma_-\rho^{-1/2}\right),
\end{aligned}
$$

where $e=-\mathcal{Q}/\mathcal{M}$; $\mathcal{M}=\rho_+$ and $\mathcal{Q}=-\sqrt{\rho_+\rho_-}$. Horowitz, Welch, 1994

It is convenient to focus on the bosonic string case, $\gamma_-=0$, $\gamma_+=\frac{1}{2}\alpha'=\frac{1}{2}\ell_s^2$.

The uncorrected (linear dilaton) black string solution is: Horne, Horowitz, Steif, 1992

$$
ds^2 = -\left(1-\frac{\rho_+}{\rho}\right)dT^2 + \left(1-\frac{\rho_-}{\rho}\right)dX^2 + \left(1-\frac{\rho_-}{\rho}\right)^{-1}\left(1-\frac{\rho_+}{\rho}\right)^{-1}\frac{\ell^2 d\rho^2}{4\rho^2}\,.
$$

## The $\gamma_\pm$-corrected T-dual black strings

with a linear dilaton, $e^{-2\Phi} = \ell\rho$, and a pure gauge KR two-form, $B_{TX} = -e$.

The corrected T-dual metric becomes diagonal:

$$\begin{aligned} ds^2 &= -\left(1-\frac{\mathcal{M}}{\rho}\right)\left(1-\frac{2\mathcal{M}}{\rho}\frac{\ell_s^2}{\ell^2}\right)dT^2 + \left(1+\frac{e\mathcal{Q}}{\rho}\right)\left(1+\frac{2e\mathcal{Q}}{\rho}\frac{\ell_s^2}{\ell^2}\right)dX^2 \\ &\quad + \left(1-\frac{\mathcal{M}}{\rho}\right)^{-1}\left(1+\frac{e\mathcal{Q}}{\rho}\right)^{-1}\frac{\ell^2 d\rho^2}{4\rho^2}\,, \\ e^{-2\Phi} &= \ell\left[\rho + \left(1+e^2\right)\mathcal{M}\,\frac{\ell_s^2}{\ell^2}\right], \\ B_{TX} &= -e - \frac{\mathcal{Q}}{\rho} + \mathcal{Q}\left(\frac{\mathcal{M}-e\mathcal{Q}}{\rho^2} - \frac{2}{\rho}\right)\frac{\ell_s^2}{\ell^2}\,, \qquad \Delta B_{TX} = \frac{\kappa\mathcal{Q}\ell_s^2}{\ell^{1/2}\rho^{3/2}}\,, \end{aligned}$$

The scalar curvature at leading order in $1/\rho$ reads

$$R = \frac{4}{\ell^2\rho}\left[(\mathcal{M}-e\mathcal{Q})\left(1+\frac{4\gamma_+}{\ell^2}\right) - 2\gamma_-\mathcal{Q}\right] \propto \frac{1}{\rho}\,,$$

which is the expected fall-off of an asymptotically flat metric in $D = 3$.

## T-duality and black hole thermodynamics

We bring our solution to Eddington-Finkelstein form:

$$dv = dt + \frac{dr}{N^2} \,, \qquad d\tilde{\chi} = d\chi - \frac{N^\chi}{N^2} dr \,,$$

in which the metric is regular at the horizon, $r = r_+$,

$$ds^2 = -N^2 dv^2 + 2dvdr + e^{-2\sigma}(d\tilde{\chi} + N^\chi dv)^2 \,.$$

The surface gravity, $\kappa$, can be defined from the timelike Killing vector,

$$\xi^M \nabla_M \xi^N \big|_{\mathcal{N}} = \kappa\, \xi^N \big|_{\mathcal{N}} \qquad \Rightarrow \qquad \kappa = \frac{r_+^2 - r_-^2}{\ell^2 r_+} = \kappa_{\mathrm{BTZ}} \,.$$

T-duality preserves the bifurcated Killing horizon and its temperature. This generalizes the HW result. Horowitz,Welch, 1994

Let us scrutinize now the properties of the black hole entropy under T-duality.

The invariance to leading order is purely a geometrical issue. Horowitz, Welch, 1994

## T-duality and black hole thermodynamics

The black hole entropy in the $\gamma_\pm$-corrected theory is not amenable to the Iyer-Wald formula because the action is not Lorentz invariant.

We must deal with the anomalous Lorentz invariance. Tachikawa, 2007

The black hole entropy is finally given by the following expression:

$$S_{BH} = 4\pi \int_\Sigma d\zeta\, e^{-2\Phi} \sqrt{\mathcal{G}_h} \left[ 1 + 2\gamma_+ \left( R^{01}{}_{01} - \frac{3}{4} H^{A01} H_{A01} \right) - 2\gamma_- \Omega^{A01} H_{A01} \right] ,$$

where $\Sigma$ denotes the cross section of the horizon and $\mathcal{G}_h$ is the determinant of the induced metric on it. As we are in 3D $\Sigma$ is a curve, parameterized by $\zeta = \psi$ in the BTZ solution or by $\zeta = \chi$ in the dual black string solution.

Plugging either the BTZ black hole or the $\gamma_\pm$-corrected black string,

$$S_{BH} = 8\pi^2 r_+ + \frac{32\pi^2}{\ell^2} \left( \gamma_+ r_+ + \gamma_- r_- \right) ,$$

where we have set $\Delta\psi = \Delta\chi = 2\pi$ at the horizon.

## T-duality and black hole thermodynamics

It is interesting to see how the expressions in the entropy integrand behave when dimensionally reduced:

$$e^{-2\Phi}\sqrt{\mathcal{G}_h} = e^{-2\phi} \ ,$$

$$R^{01}{}_{01} - \frac{3}{4}H^{012}H_{012} = \tilde{R}^{01}{}_{01} - \frac{3}{4}\left(e^{2\sigma}V^{01}V_{01} + e^{-2\sigma}W^{01}W_{01}\right) \ ,$$

$$H^{201}\Omega_{201} = -\frac{1}{2}W^{01}V_{01} \ ,$$

where $\tilde{R}^{\mu\nu}{}_{\rho\sigma}$ is the dimensionally reduced Riemann tensor, and $V_{\mu\nu}$ and $W_{\mu\nu}$ are the field strengths corresponding, respectively, to $V_\mu$ and $W_\mu$.

Each of the terms is T-dual invariant by itself; the latter two are only requested to be invariant under the uncorrected Buscher rules.

Notice that for $\gamma_- \neq 0$ the entropy depends on $r_-$. This is standard in theories with broken parity. Couso-Santamaria, Edelstein, Garbarz, Giribet, 2011

## AdS/CFT constraints

The entropy of the BTZ solution matches the Cardy formula. The central charges of the dual CFT can be readily computed,

$$c_{\pm} = \frac{3\ell}{2G_N} + \frac{6}{\ell G_N}(\gamma_+ \pm \gamma_-) \, ,$$

where we have reintroduced $16\pi G_N$ for the sake of comparison. In the case $\gamma_\pm = 0$ we recover the Brown-Henneaux seminal result, Brown, Henneaux, 1986

$$c_{\pm} = \frac{3\ell}{2G_N} \, .$$

These expressions satisfy the integrated version of the Cardy formula,

$$S_{BH} = \frac{\pi^2 \ell}{3}(c_+ T_+ + c_- T_-) \, , \qquad \text{where} \quad T_{\pm} = \frac{r_+ \pm r_-}{2\pi\ell^2} \, .$$

The CFT description, being non-symmetric in its left- and right-moving sectors due to the bulk CS term, yields a quantization condition for $\gamma_-$.

Modular invariance imposes $c_- - c_+ = 24k$, $k \in \mathbb{Z}$. Witten, 2007

$\gamma_-$ must be quantized in units of $2\ell G_N$ ($G_N \ll \Lambda_\star^{-1} \ll \ell$, $\ell$ is the AdS radius).

## Concluding remarks

We applied T-duality to a BTZ black hole in a family of higher-derivative theories which includes (but generalizes) the $\alpha'$-corrected string actions.

We used T-duality as a solution generating technique to obtain an approximate non-trivial higher-curvature asymptotically flat black string configuration.

The temperature and the entropy remain invariant for all values of $\gamma_\pm$.

The result is puzzling: the theories we have discussed, for generic $\gamma_\pm$ do not seem to have a sigma model origin. T-duality as an approximate symmetry?

Bifurcate Killing horizons are mapped onto bifurcated Killing horizons with exactly the same surface gravity, generalizing earlier results. Horowitz, Welch, 1994

The preservation of entropy, in turn, is remarkable. Here it is not just area conservation.

One might naively expect that neglecting quadratic contributions in $\gamma_\pm$ may affect a quantity that accounts for the number of degrees of freedom.

## Concluding remarks

Our results suggest that T-duality could be a symmetry of a larger class of theories.

Among all, what makes string theory special is probably the fact that it can be resummed to all orders in $\gamma_\pm \sim \alpha'$.

Our results are in line with expectations arising from DFT. Arvanitakis, Blair, 2017

The invariance of $\gamma_\pm$-corrected temperature and entropy can be generalized to black holes in higher dimensions. Edelstein, Sfetsos, Sierra-García, Vilar López, to appear

A further use of AdS/CFT to constrain $\gamma_\pm$ leading to consistent quantum gravities in three dimensions must be done. Edelstein, Giribet, Sierra-García, to appear

There is a chiral point where one of the central charges vanishes and a logarithmic branch emerges, much as in the case of TMG.

# Thank you