Aspects of Gravity in $D \neq 4$

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Based on 1805.12160 and 1806.06254, in collaboration with G. Giribet, E. Lavia N. Grandi and J. Oliva.

Plan for the talk

1. Motivation

2. Exotic 3D Massive Gravity

3. Q-curvature and Gravity

4. Final remarks

Motivation

Why should we study 3D gravity?

→ 3D gravity shares many interesting features with its four dimensional analog, for example, the existence of black holes (BTZ).

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→ 3D gravity shares many interesting features with its four dimensional analog, for example, the existence of black holes (BTZ).

→ 3D gravity it is much more accessible so it is a perfect toy model to explore theoretical aspects of gravity.

Examples:

- Microscopic description on non-susy BH entropy.
- Sum over geometries and study the quantum gravity partition function.
- Investigate the consistency of holography (2D CFT).

Motivation

Why should we study 3D gravity?

- Einstein gravity in 3D does not have local degrees of freedom which reduces the content of the theory to BH and massless boundary gravitons (but no propagating modes).

- The theory can acquire propagating modes when one deforms it by adding mass to the graviton (in 3D there exist consistent ways of giving mass to the graviton).

Let us visit some of these roads…

Motivation

Why should we study 3D gravity?

Topologically Massive Gravity with negative cosmological constant

[Deser et al.]

The action is given by

$$I = \frac{1}{16\pi G}\int d^3x\sqrt{-g}(R-2\Lambda) + \frac{1}{32\pi G\mu}\underbrace{\int d^3x\sqrt{-g}\epsilon^{\lambda\mu\nu}\Gamma^{\rho}_{\lambda\sigma}[\partial_\mu\Gamma^{\sigma}_{\rho\nu} + \frac{2}{3}\Gamma^{\sigma}_{\mu\tau}\Gamma^{\tau}_{\nu\rho}]}_{\text{Gravitational Chern-Simons action}}$$

With $\Lambda = -1/l^2$ and μ a mass parameter (graviton mass).

$\boxed{\text{The eom are 3rd order.}}$

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With $\Lambda = -1/l^2$ and μ a mass parameter (graviton mass).

→ For every value of μ TMG has an AdS_3 as solution with radius l

→ For large l, the linearized excitations about AdS_3 describe a propagating graviton with mass μ but negative energy!. Then AdS_3 is generically unstable.

Motivation

Why should we study 3D gravity?

Topologically Massive Gravity with negative cosmological constant

→ Brown and Henneaux showed that quantum gravity on asymptotically AdS_3 spacetimes with certain BC is described by a 2D CFT which lives on the boundary.

→ For TMG the two central charges are given by

$$c_L = \frac{3l}{2G}(1 - \frac{1}{\mu l}) \qquad \text{And} \qquad c_R = \frac{3l}{2G}(1 + \frac{1}{\mu l})$$

[Kraus et al.]

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- For $\mu l = 1$, AdS_3 (and BTZ) are presumably stable under small perturbations.This is call Chiral gravity ($c_L = 0$ and $c_R = 3l/G$). [Strominger et al.]
- For $\mu l > 3$, there are warped AdS_3 (and warped AdS_3 BH) solutions that are presumably stable (these are NON-Einstein solutions). [Strominger et al.]

Motivation

Why should we study 3D gravity?

New Massive Gravity with negative cosmological constant

[Townsend et al.]

The action is given by

$$I = \frac{1}{16\pi G}\int d^3x\sqrt{-g}(R-2\Lambda) + \frac{1}{16\pi G m^2}\int d^3x\sqrt{-g}(R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2)$$

Where m represents the mass of the graviton. The eom are 4th order.

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This coefficient is essential for the theory to be ghost free about sensible backgrounds.

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→ NMG has a large set of interesting solutions, e.g. AdS_3 , Warped AdS_3 , Lifshitz spaces, etc. (i.e. Einstein and non Einstein)

→ For AdS_3 the corresponding central charge is given by

$$c = \frac{3l}{2G}(1 + \frac{1}{2m^2 l^2})$$

[Liu et al.]

Motivation

Why should we study 3D gravity?

Both, TMG and NMG present a consistency problem when discussed in the context of AdS/CFT. This problem is known as the bulk-boundary clash. It means that there is no way to achieve a unitary theory in the bulk and the boundary at the same time.

Bulk graviton with positive energy $\longleftrightarrow$ Negative central charge

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Bulk graviton with positive energy $\longleftrightarrow$ Negative central charge

There are different attempts to solve this problem. The most recent is

$$\boxed{\text{Exotic 3D massive gravity}}$$

Exotic 3D Massive Gravity

[Townsend et al.]

It is defined by the following fourth-order equations of motion,

$$\underbrace{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu}}_{\text{Einstein gravity}} + \underbrace{\frac{1}{\mu} C_{\mu\nu}}_{\text{TMG term}} \underbrace{- \frac{1}{m^2} H_{\mu\nu} + \frac{1}{m^4} L_{\mu\nu}}_{\text{Exotic term}} = 0$$

Where

$$C_{\mu\nu} = \frac{1}{2} \epsilon_\mu{}^{\alpha\beta} \nabla_\alpha R_{\beta\nu} + \frac{1}{2} \epsilon_\nu{}^{\alpha\beta} \nabla_\alpha R_{\beta\mu} \ , \quad H_{\mu\nu} = \epsilon_\mu{}^{\alpha\beta} \nabla_\alpha C_{\nu\beta} \ , \quad L_{\mu\nu} = \frac{1}{2} \epsilon_\mu{}^{\alpha\beta} \epsilon_\nu{}^{\gamma\sigma} C_{\alpha\gamma} C_{\beta\sigma} .$$

And $C_{\mu\nu}$ is the Cotton tensor. The limit $m \to \infty$ corresponds to TMG.

IMPORTANT: The equations of motion do not come from varying any action!
The Bianchi identities are only satisfy on-shell.

Exotic 3D Massive Gravity

[Townsend et al.]

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Where

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And $C_{\mu\nu}$ is the Cotton tensor. The limit $m \to \infty$ corresponds to TMG.

→ In D=3, all Einstein manifolds are conformally flat $\leftrightarrow$ $C_{\mu\nu} \equiv 0$
Then, GR appears as a sub-sector of the space of solutions of EMG.

→ EMG also admits non-Einstein solutions.

Exotic 3D Massive Gravity

Vacua of EMG (a quick glimpse)

Let us focus on solutions to the eom when the parameters satisfy,

$$\boxed{\mu = \frac{m^2 l}{1 - m^2 l^2}, \qquad l^2 = -1/\Lambda}$$

We will call this the CRITICAL POINT.

Exotic 3D Massive Gravity

Vacua of EMG (a quick glimpse) [Chernicoff et al.]

Let us focus on solutions to the eom when the parameters satisfy,

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An important example: at the critical point the following non-Einstein metric is a solution of the eom,

$$ds^2 = -r^2 dt^2 + \frac{dr^2}{r^2} + r^2 d\phi^2 + N_\gamma(t,r)(dt + d\phi)^2 \quad \text{with} \quad N_\gamma(t,r) = \left(\beta(t - t_0) + \frac{\gamma}{r^4}\right)$$

And is asymptotically AdS_3 in the Brown-Henneaux sense, i.e.

$$g_{tt} \simeq r^2 + \mathcal{O}(1)\ , \quad g_{t\phi} \simeq \mathcal{O}(1)\ , \quad g_{\phi\phi} \simeq r^2 + \mathcal{O}(1)$$

$$g_{rr} \simeq r^{-2} + \mathcal{O}(r^{-4})\ , \quad g_{tr} \simeq \mathcal{O}(r^{-4})\ , \quad g_{r\phi} \simeq \mathcal{O}(r^{-4})$$

Exotic 3D Massive Gravity

Vacua of EMG (a quick glimpse)

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→ The solution exhibits time translation symmetry even though explicitly depends on t.

→ In the case $\beta = 0$, where $N_\gamma = 0$, the metric reduces to that of massless BTZ.

→ For $\beta \neq 0$, the metric is NOT conformally flat!

This example is important because it shows that the Birkhoff theorem does not hold in Exotic Massive Gravity.

Exotic 3D Massive Gravity

Vacua of EMG (a quick glimpse)

Besides those of GR and the example that I just showed, EMG admits other interesting type of solutions which appear on the curve of the parameter space (μ, m).

- Warped AdS_3
- Lifshitz black holes
- Euclidean vacua

The first two specially important in the context of AdS/CFT.

Now, let us move from $D = 3$

Q-curvature and Gravity

Why should we study gravity in D>4?

→ On general grounds, for $D \geq 4$ higher-curvature modifications render the theory of gravity renormalizable, but at the cost of introducing ghost instabilities and other pathologies.

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→ Some known examples that are well behaved:

- In $D = 4$, Critical Gravity, which is defined by

Coupling constant

$$I = \frac{1}{\kappa^2}\int d^4x \underbrace{\sqrt{-g}(R - 2\Lambda}_{\text{EH term}} + L^2 \underbrace{C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu}}_{\text{Conformally invariant linear combination of } R^2 \text{ terms.}})$$

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Coupling constant

This theory includes GR, there no ghost around AdS and has a second massless spin-2 field.

In $D > 4$ CG can be defined but generically does not admit Einstein spaces.

Q-curvature and Gravity

Why should we study gravity in D>4?

- On general grounds, for $D \geq 4$ higher-curvature modifications render the theory of gravity renormalizable, but at the cost of introducing ghost instabilities and other pathologies.

- Some known examples that are well behaved:

  - In $D > 4$, Lovelock theory which is defined by dimensionally extending topological invariants, e.g

$$I = \frac{1}{\kappa^2}\int d^5x\sqrt{-g}\Big[R + 2\Lambda + \alpha\underbrace{\Big(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}\Big)}_{\text{Only topologically invariant term}}\Big]$$

Second order eom, no ghost around AdS, etc.

Q-curvature and Gravity

The family of theories we want to explore can be thought of as a hybrid between CG and Lovelock, in the sense that are defined by dimensionally extending conformal invariants, in opposition to topological invariants.

The fundamental building block to construct the action of the theory is the so called Q-curvature. Let us motivate its definition…

Warning: what follows is a bit technical but the end result is quite simple.

Q-curvature and Gravity

Given the Weyl rescaling of a D-dimensional metric $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = e^{2\varphi} g_{\mu\nu}$, we consider linear differential operator $P_{m,n}$ (with $m \in 2\mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{\geq 0}$), that transform covariantly as

$$\tilde{P}_{m,n}(f) = e^{-\frac{n+m}{2}\varphi} P_{m,n}(e^{\frac{n-m}{2}\varphi} f)$$

This operator has the form,

$$P_{m,n} = \Delta_{m,n} + \frac{n-m}{2} Q_{m,n} \ , \qquad \Delta_{m,n} = \underbrace{\Box^{\frac{m}{2}}}_{\text{Laplace operator}} + \underbrace{\ldots}_{\text{Constant terms}}$$

$Q_{m,n}$ is called the n-dimensional Q-curvature of m-order. Under Weyl, transform as

$$\tilde{Q}_{m,n} = e^{-\frac{n+m}{2}\varphi} \left(Q_{m,n} + \frac{2}{n-m} \Delta_{m,n} \right) e^{\frac{n-m}{2}\varphi}$$

Q-curvature and Gravity

Examples:

$m = 2$ (arbitrary spacetime dimensions)

$$\underbrace{Q_{2,n}}_{\text{Gaussian curvature}} = -\frac{1}{2(n-1)} R \,, \quad \underbrace{P_{2,n}}_{\text{Yamabe operator}} = \Box + \frac{n-2}{2} Q_{2,n} \,, \quad \Delta_{2,n} = \Box .$$

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$m = 4$ (arbitrary spacetime dimensions)

$$\underbrace{Q_{4,n}}_{\text{Branson's Q-curvature}} = -\frac{1}{2(n-1)} \Box R - \frac{2}{(n-2)^2} R_{\mu\nu} R^{\mu\nu} + \frac{n^2(n-4) + 16(n-1)}{8(n-1)^2(n-2)^2} R^2 ,$$

$P_{4,n}$ is called the Paneitz operator (but I won't write it down).

[Fradkin and Tseytlin]

And all these for what???

Q-curvature and Gravity

Connection between Q-curvature and conformal invariants…

According to the Gauss-Bonnet theorem, in dimension 2,

$$\mathcal{I} = -\frac{1}{2\pi}\int_{M_2} d^2x\sqrt{g}\,Q_{2,2} = \frac{1}{4\pi}\int_{M_2} d^2x\sqrt{g}R = \underbrace{\chi(M_2)}_{\text{Topological invariant}}$$

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And we know that all metrics are locally conformally equivalent. We also have the following properties:

If $g_{\mu\nu} \to e^{2\varphi} g_{\mu\nu}$,then $R \to e^{-2\varphi}(R - 2\Delta_{2,2}\varphi)$ and $\Delta_{2,2} \to e^{-2\varphi}\Delta_{2,2}$

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In dimension 4,

$$\mathcal{I} = \frac{1}{8\pi^2}\int_{M_4} d^4x\sqrt{g}\,Q_{4,4} + \frac{1}{32\pi^2}\int_{M_4} d^4x\sqrt{g}\,C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta} = \chi(M_4)$$

Let us see what exactly are we doing here…

Q-curvature and Gravity

Connection between Q-curvature and conformal invariants…

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$Q_{4,4}$ is the fourth order, four dimensional curvature invariant that under Weyl transform as $Q_{4,4} \to e^{-4\varphi}(\Delta_{4,4}\varphi + Q_{4,4})$

Explicitly, is given by $Q_{4,4} = -\frac{1}{6}\Box R - \frac{1}{2}R_{\mu\nu}R^{\mu\nu} + \frac{1}{6}R^2$

Q-curvature and Gravity

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Both terms in the action are conformal invariants. That is, the Q-curvature computes a topological invariant within a given conformal class.

Mr. Branson gave us a definition of the Q-curvature in arbitrary dimension.

Q-curvature and Gravity

The gravity action we will consider is defined by the sum of the dimensionally continued conformal invariants; namely

$$\mathcal{I} = \frac{1}{16\pi G}\int_{M_n} d^n x\sqrt{-g}\Big(R - 2\Lambda + \frac{4L^2}{(n-2)^2(n-4)}\Big(R_{\mu\nu}R^{\mu\nu} - \frac{n^3-4n^2+16n-16}{16(n-1)^2}R^2\Big) + ...\Big)$$

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This action preserves Einstein spaces as solutions and includes some of the results present in the literature (e.g. conformal gravity, chiral gravity).

The theory described by this action has among its solutions

- Black holes
- Non- Einstein metrics
- Gravitational waves

All the details can be found in **1805.12160.**

Final remarks

- There is a lot that we can learn by studying aspects of gravity in different dimensions.
- 3D gravity is probably the best toy model to start understanding quantum gravity.
- Close connection with different topics of AdS/CFT.
- BH with very interesting thermodynamics (did not mentioned anything in this talk).

And…

Final remarks

This is Olivia! My newly born…