LIVING ON THE WEDGE: Entanglement Wedge Reconstruction and Entanglement of Purification

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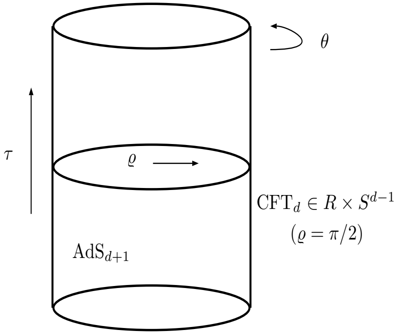
Based on 1804.05855 w/ A. Güijosa, A. Landetta J.F. Pedraza
and 1708.02958 w/ A. Güijosa and J.F. Pedraza

Motivation for reconstruction

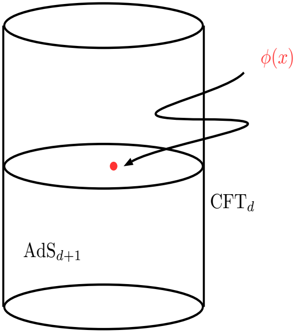
- Within the AdS/CFT framework, it's natural to ask how does the dynamical space-time emerge from the d.o.f. living on the lower-dimensional theory?
- The duality suggest a deep connection between the emergence of dynamical spacetime and the entanglement pattern of a large number of d.o.f. at the boundary.



- 'Hole-ography' (or differential entropy) is an important step towards holographic reconstruction.

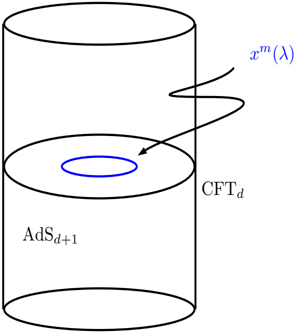
Two approaches for reconstruction

- Bulk reconstruction of LOCAL operators.



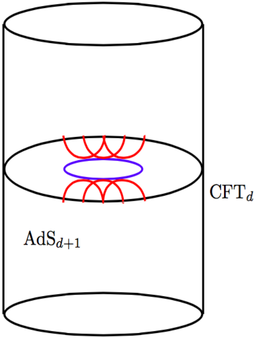
Two approaches for reconstruction

- Bulk reconstruction of LOCAL operators.
- Reconstruction of spacelike curves.



Two approaches for reconstruction

- Bulk reconstruction of LOCAL operators.
- Reconstruction of spacelike curves.
- We study the second approach using 'differential entropy', which is a combination of entanglement entropies.

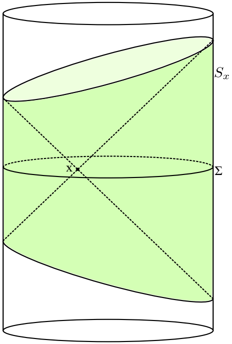


Bulk reconstruction of local operators

- It's possible to find a representation of the bulk field $\phi(\varrho, \tau, \theta)$ in terms of CFT data, i.e., to accomplish global reconstruction:

$$\phi(x) = \int_{S_x} dX \; K(x;X)\mathcal{O}(X) \; ,$$

where x and X denote bulk and boundary coordinates respectively, $K(x;X)$ is the smearing function, i.e., the kernel we smear against the boundary operator $\mathcal{O}$ to generate ϕ.



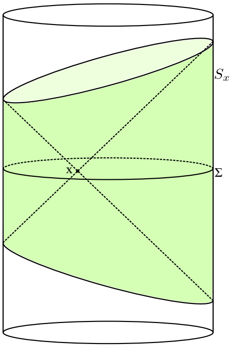
[Banks,Douglas, Horowitz, Martinec; Hamilton Kabat, Lifschytz, Lowe; etc]

Bulk reconstruction of local operators

$$\phi(x) = \int_{S_x} dX \; K(x;X)\mathcal{O}(X) \; .$$

- Non-local CFT representation in space and time.
- This particular representation uses CFT d.o.f. on an entire Cauchy slice Σ.
- We'll focus on AdS_3/CFT_2 case from now on for simplicity.

  [Banks,Douglas, Horowitz, Martinec; Hamilton Kabat, Lifschytz, Lowe; etc]

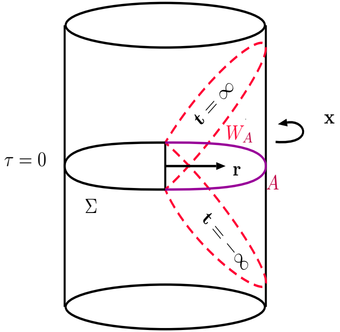


Rindler Wedge

- It's possible to compute a different representation with spatial support on a subregion, $A \subset \Sigma$.
- E.g. taking A as a boundary semicircle, the following coordinates define the Rindler wedge W_A

$$ds^2 = -\mathbf{r}^2 d\mathbf{t}^2 + (1+\mathbf{r}^2) d\mathbf{x}^2 + \frac{d\mathbf{r}^2}{1+\mathbf{r}^2} \ ,$$

where $\mathbf{r} = \sqrt{\varrho^2 - 1}$, $\mathbf{r} \in (0, \infty)$ and $\mathbf{t}, \mathbf{x} \in (-\infty, \infty)$.



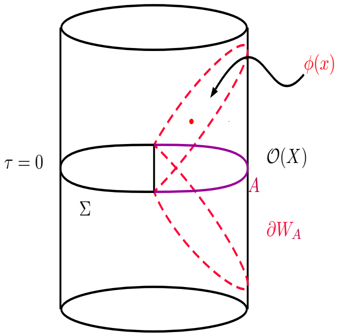
Rindler Reconstruction

- If $\phi \in W_A$, we can write down the following formula in terms of CFT$_2$ data

$$\phi(x)|_{x \in W_A} = \int_{\partial W_A} dX \; K(x, X) \mathcal{O}(X) \; .$$

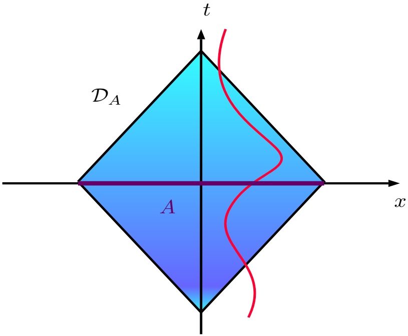
- ∂W_A is equivalent to the boundary domain of dependence (or causal diamond) of A, $\mathcal{D}_A$.

  [Hamilton Kabat, Lifschytz, Lowe; Morrison]



Domain of Dependence

Given a boundary spatial region A, the domain of dependence $\mathcal{D}_A$ is defined as the set of boundary points such that any boundary causal curve passing through one of these points must also intersect A.

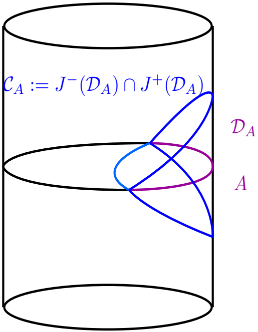


Causal Wedge Reconstruction

- A bulk region can be associated with $\mathcal{D}_A$, the causal wedge of A, $\mathcal{C}_A$. It's defined as the intersection of the bulk future and bulk past of $\mathcal{D}_A$

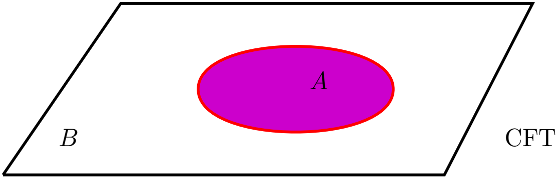
$$\mathcal{C}_A := J^+[\mathcal{D}_A] \cap J^-[\mathcal{D}_A] \ .$$

- Points inside $\mathcal{C}_A$ are causally connected with $\mathcal{D}_A$.
- The boundary region A has complete information about what's going on in $\mathcal{C}_A$. [Hamilton Kabat, Lifschytz, Lowe; Morrison; Van Raamsdonk; Hubeny, Rangamani]



Entanglement Entropy

- Consider a CFT in a state $|\psi\rangle$, and a spatial region at fixed time τ_0. Dividing it into two regions A and B, we define the reduced density matrix associated to A as $\rho_A = \mathrm{Tr}_B |\psi\rangle\langle\psi|$.
- The entanglement entropy associated with A is then the Von Neumann entropy $S_A = -\mathrm{Tr}\Big(\rho_A \log \rho_A\Big)$.
- In general, it's technically very complicated to compute, even in free theories!



Holographic Entanglement Entropy

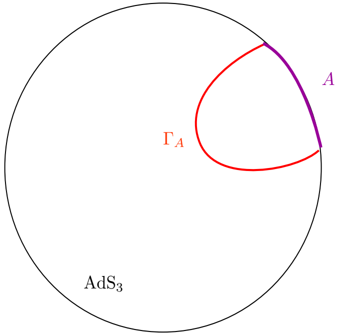
- The RT proposal for computing the EE associated to A states

$$S_A = \frac{\mathcal{A}(\Gamma_A)}{4G_N} \ ,$$

$\mathcal{A}(\Gamma_A)$ is the area of thegeodesic Γ_A that is homologous to A (w/ $\partial\Gamma_A = \partial A$).

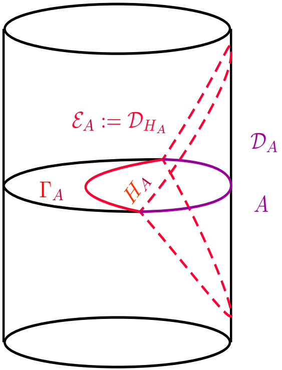
- It's a useful tool for computing EE in strongly coupled field theories.
- Viceversa, if we know the pattern of entanglement, we can translate it to minimal bulk areas anchored to the boundary.

[Ryu,Takayanagi;Hubeny,Rangamani,Takayanagi;Lewkowycz, Maldacena;Dong,Lewkowycz,Rangamani]



Entanglement Wedge

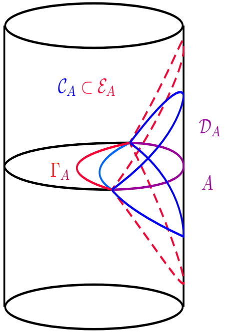
- The RT surface defines a subregion in the bulk associated with the interval A called 'Entanglement Wedge', $\mathcal{E}_A$.
- It's defined as the bulk domain of dependence $\mathcal{D}_{H_A}$ of any homology hypersurface H_A for the RT surface Γ_A. [Headrick, Hubeny, Lawrence, Rangamani]



Importantly, in empty AdS the entanglement wedge of a ball-shaped boundary region is equivalent to its AdS-Rindler wedge, $W_A = \mathcal{E}_A$.

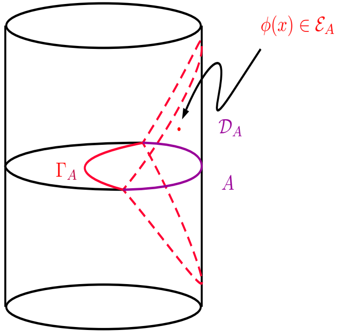
Causal and Entanglement Wedges

- In general, the entanglement wedge can be quite bigger than the causal wedge
- The crucial point is that the causal wedge of any boundary region can never contain points behind the event horizon, while the entanglement wedge can.
- In fact, under 'reasonable assumptions' we have the containment: $\mathcal{C}_A \subset \mathcal{E}_A$.
  [Wall; Headrick, Hubeny, Lawrence, Rangamani]



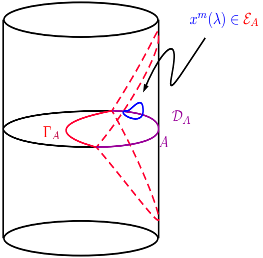
Entanglement Wedge Reconstruction

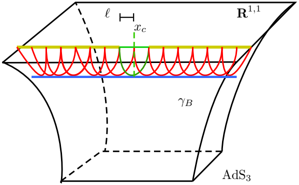
- Subregion duality $\mathcal{E}_A \leftrightarrow \mathcal{D}_A$.
- This was proved for local bulk operators living on the entanglement wedge using the quantum error correction interpretation of holography. [Dong, Harlow & Wall]



Bulk Reconstruction of Curves

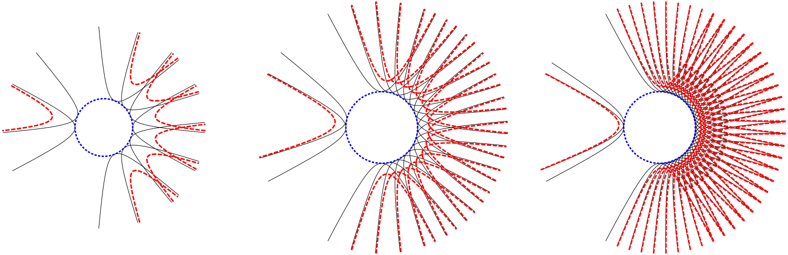
- We'll explore a different but related aspect of bulk reconstruction.
- Main motivation: Reconstruction of spacelike curves inside the entanglement wedge.
- Our main tool will be Hole-ography or differential entropy for computing lengths of curves inside the wedge.
- In the general case, we overcome a challenge for reconstruction using two concepts 'null alignment' and entanglement of purification.





Hole-ography

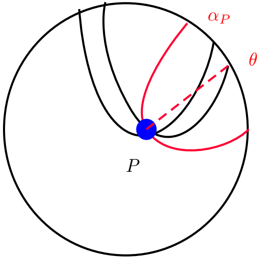
- We can compute the length of a given spacelike curve by adding and subtracting geodesics tangent to the curve.
- Therefore, spacelike curves can be represented by families of intervals or geodesics.
- Length encoded in 'differential entropy' formula
  $E[\alpha(\theta)] = \int d\theta \frac{dS(\alpha)}{\alpha}\Big|_{\alpha=\alpha(\theta)} = \frac{2\pi R_0}{4G_N}$.

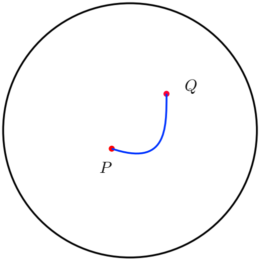


[Balasubramanian, Chowdhury, Czech, de Boer, Heller; Myers, Rao, Sugishita; Czech, Dong, Sully; Czech, Lamprou, McCandlish, Sully; Headrick, Myers, Wien]

Bulk Geometry Reconstruction

- Using this approach it's possible to reconstruct the most basic ingredients of the bulk geometry [Czech, Lamprou].
- By shrinking closed curves to zero size, it's possible to define points in the bulk. Given two such points, we can compute the distance between them in terms of the entanglement pattern at the boundary.
- Hole-ography thus provides direct access to the spacetime on which local operators $\phi(x)$ are placed, and therefore conceptually underlies the local operators bulk reconstruction discussed before.





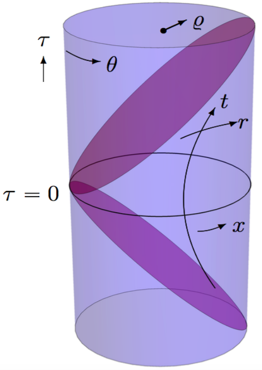
Poincaré AdS

- The simplest example of an entanglement wedge: the Poincaré patch.
- It's obtained by defining A to be the whole Cauchy slice but without a single point.

$$ds^2 = \frac{L}{z^2}(-dt^2 + dx^2 + dz^2) \ ,$$

$t, x \in (-\infty, \infty)$ and $z \in [0, \infty)$.

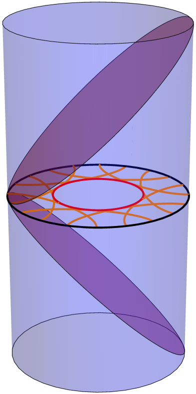
- Physically, these coordinates are associated with a family of uniformly accelerated observers ($a = 1/L$). The horizon ($z \to \infty$) determines the boundary of the region in which can causally interact.



Hole-ography in Poincaré

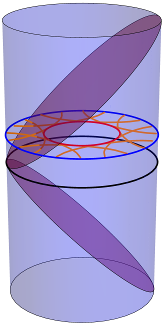
- Consider a circle located at $t = 0$. Both $\tau = 0$ and $t = 0$ slices coincide.
- Then, all tangent geodesics needed for reconstruction are fully contained within the Poincaré wedge.
- In fact, by t-invariance of the metric reconstruction is accomplished in any fixed t slice.

$$ds^2 = \frac{L}{z^2}(-dt^2 + dx^2 + dz^2) \ ,$$



Challenge to Hole-ography

- However, the story is different when $\tau \neq 0$, as the global slice lies partially outside the Poincaré wedge.
- Some of the geodesics then cross the horizon and aren't suitable for reconstruction. They can't be interpreted as entanglement in the CFT.
- In general, spacelike curves have nonreconstructible segments.
- This represents a challenge to the standard hole-ographic procedure.

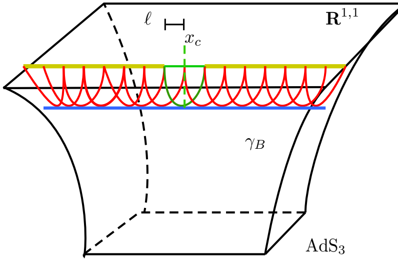


Hole-ographic reconstruction

- Hole-ography gives a natural map between spacelike curves and families of intervals (or geodesics) at the boundary.
- Given a curve, we can associate a family of tangent geodesics attached to the CFT, whose endpoints $x_{\pm}$ define the boundary of the intervals.

$$E := \oint d\lambda \left(\frac{\partial S(x_{-}(\lambda), x_{+}(\lambda'))}{\partial \lambda'} \right)\Big|_{\lambda'=\lambda} .$$

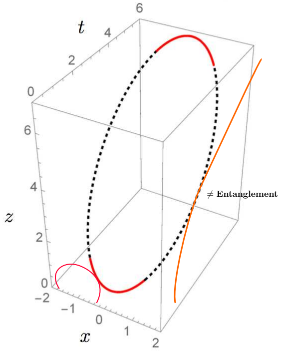
[Balasubramanian et al.; Myers et al.; Headrick et al.]



Failure of reconstruction

- It's not possible, in general, to associate families of intervals/geodesics to a given bulk curve.
- In fact, a segment along the curve is reconstructible if the projection of its tangent vector, u, satisfies

$$-(u^t)^2 + (u^x)^2 > 0\ , \quad \text{i.e is spacelike}\ .$$

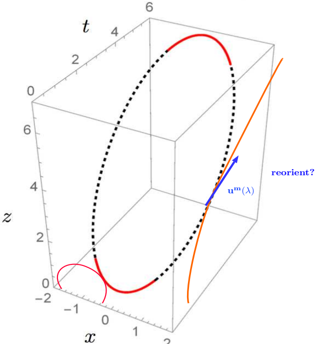


'Null Alignment'

In a different context, [Headrick, Myers, Wien] found that differential entropy formula reproduces lengths of curves correctly if we choose non-standard families of intervals/geodesics obtained by translating its tangent vector by a null and orthogonal vector n

$$u \to U := u + n \ , \quad n \cdot n = 0 \text{ and } n \cdot u = 0 \ .$$

Can we use this freedom to reorient tangent geodesics on nonreconstructible segments to end on the boundary?

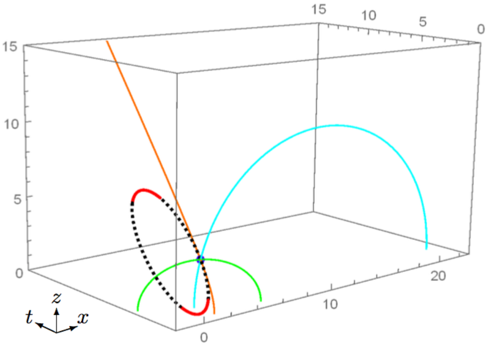


Reconstruction Achieved

We find that 'null alignment' always allows us to shoot geodesics that do reach the Poincaré boundary if

$$-(U^t)^2 + (U^x)^2 > 0 \;\leftrightarrow\; (u^z + n^z)^2 < (z^2/L^2)u \cdot u \ .$$

Thus, any spacelike curve within the Poincaré wedge can be fully reconstructed. Moreover, there are infinitely many ways to represent a bulk curve in terms of boundary intervals.

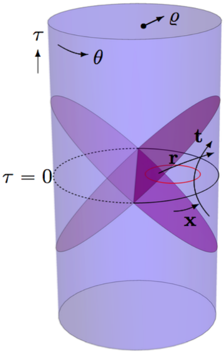


Rindler Wedge Hole-ography

- Let's move on to a smaller entanglement wedge.
- Rindler Wedge:

$$ds^2 = -\mathbf{r}^2 d\mathbf{t}^2 + (1+\mathbf{r}^2)d\mathbf{x}^2 + \frac{d\mathbf{r}^2}{1+\mathbf{r}^2}\ .$$

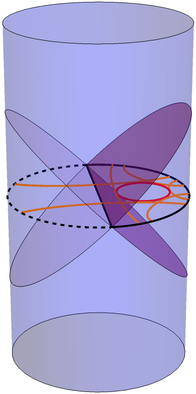
- Can we accomplish hole-ographic reconstruction?



Rindler Wedge Hole-ography

- The problem is more complicated, as even in the static case there are geodesics exiting the wedge.
- Is 'null alignment' sufficient to accomplish reconstruction of arbitrary spacelike curves?

$$u \to U := u + n \ , n \cdot n = 0 \text{ and } n \cdot u = 0 \ .$$



Failure of Reconstruction

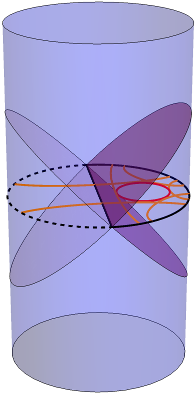
- In this case, we must fulfill two conditions:

$$(U^x)^2 - (U^t)^2 > 0$$

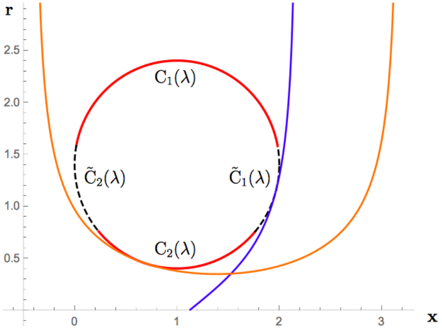
$$\mathbf{r}^2(1+\mathbf{r}^2)(U^{\mathbf{x}} + U^{\mathbf{t}})^2 - (U^{\mathbf{r}})^2 > 0$$

$$\mathbf{r}^2(1+\mathbf{r}^2)(U^{\mathbf{x}} - U^{\mathbf{t}})^2 - (U^{\mathbf{r}})^2 > 0$$

- The second inequality implies the third, or vice-versa, depending on the relative sign of $U^{\mathbf{x}}$ or $U^{\mathbf{t}}$.
- We find null alignment condition allows us to satisfy only one of these inequalities, but not both at the same time.



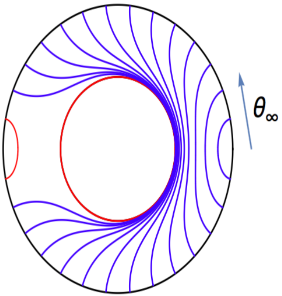
E.g., consider a circle at constant Rindler time. Even after null alignment, it cannot be reconstructed on the sides.

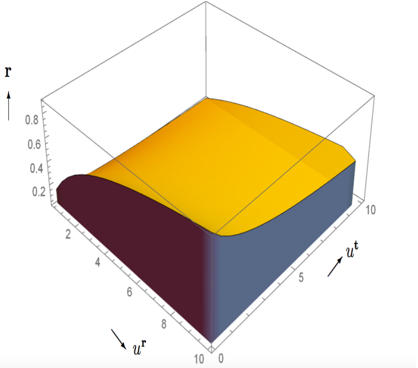


The crucial feature is that at any given radial depth **r**, sufficiently steep tangent vectors u are such that no allowed $U = u + n$ corresponds to a boundary-anchored geodesic.

Entanglement Shade

- Similar concept to the phenomenon of "entanglement shadows": bulk regions which are not reached by geodesics. [Hubeny, Maxfield, Rangamani, Tonni; Engelhardt, Wall; Balasubramanian et al.; Freivogel, Jefferson, Kabir, Mosk, Yang; Engelhardt, Fischetti].
- Entanglement shade: entanglement shadow in the tangent bundle of the spacetime.





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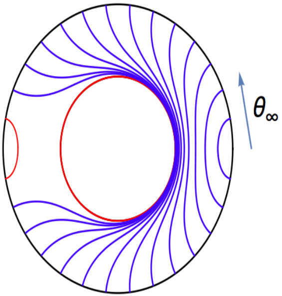
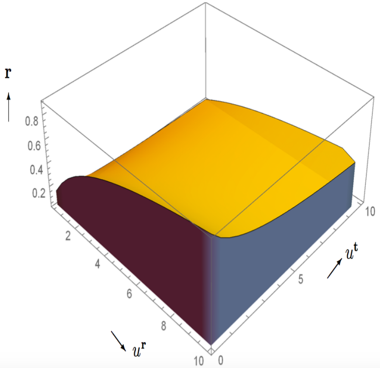
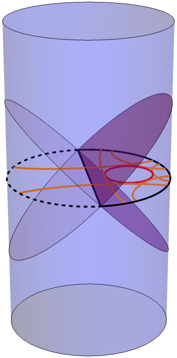
- The entanglement shadow is a well-delineated region of spacetime, whereas entanglement shade is defined on its tangent bundle.

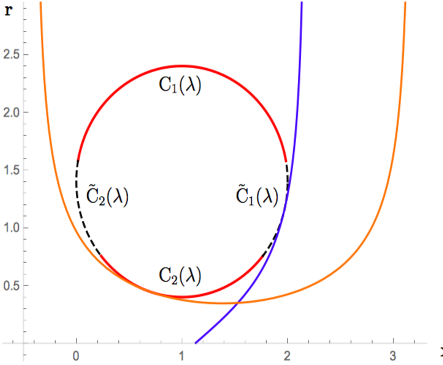




Accomplish reconstruction?

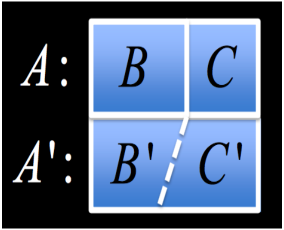
- In order to accomplish reconstruction of arbitrary spacelike curves, we need to use the set of tangent geodesics that are not boundary-anchored.
- Such geodesics are not associated with entanglement entropies. Do they have some interpretation in the CFT?





Entanglement of Purification

- Let the bipartitioned system $A = B \cup C := BC$ be in a state described by a density matrix ρ_{BC}.
- For a mixed state the entanglement entropy $S_{BC} > 0$. In this case $S_B \neq S_C$, it quantifies quantum and classical correlations.

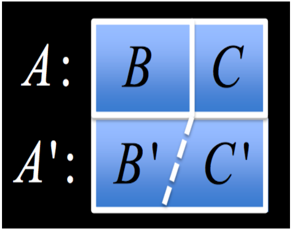


Entanglement of Purification

- A purification of this system is a set A' of additional d.o.f. and a choice of pure state $|\psi\rangle$ for the overall system $A'A$ such that

$$\rho_A = \mathrm{Tr}_{A'} |\psi\rangle\langle\psi| \ .$$

- S_A then comes from the entanglement between A and A'.
- Partitioning the auxiliary system $A' = B'C'$ once again, we can compute the entropies $S_{BB'} = S_{CC'}$.

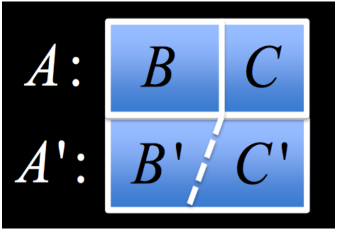


Entanglement of Purification

- Of course, there are many ways in which we can purify a system.
- By optimizing among all possible purifications and partitions $B'C'$, the entanglement of purification between B and C is defined as

$$P(B:C) := \min_{|\psi\rangle, B'} S_{BB'} \; .$$

[Terhal,Horodecki,Leung, Di Vicenzo]

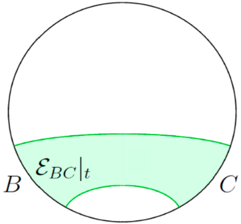


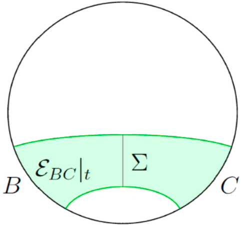
Holographic Entanglement of Purification

- It was recently proposed that the entanglement wedge cross section is dual to the entanglement of purification

$$P(B:C) = \min \frac{\mathcal{A}(\Sigma')}{4G_N} \ ,$$

where Σ' is the geodesic ending on Γ_{BC} and divides B and C.
[Takayanagi, Umemoto; Nguyen,Devakul,Halbash,Zaletel,Swingle]





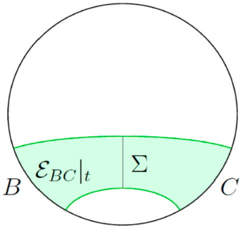
Holographic Entanglement of Purification

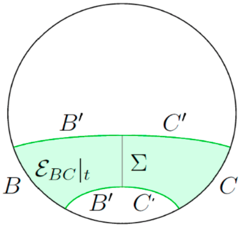
- It was recently proposed that the entanglement wedge cross section is dual to the entanglement of purification

$$P(B:C) = \min \frac{\mathcal{A}(\Sigma')}{4G_N} \ ,$$

where Σ' is the geodesic ending on Γ_{BC} and divides B and C.

- Both quantities satisfy the same set of inequalities! [Takayanagi, Umemoto; Nguyen,Devakul,Halbash,Zaletel,Swingle]

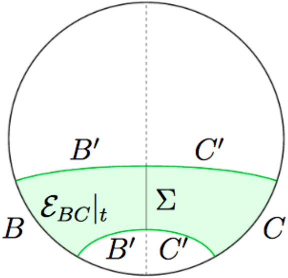




Holographic Entanglement of Purification

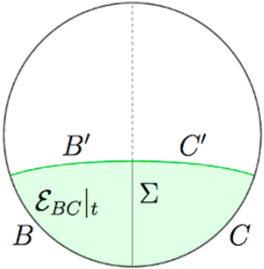
$$P(B:C) = \min \frac{\mathcal{A}(\Sigma)}{4G_N} \ ,$$

- Importantly, this recipe uses truncated geodesics, so the optimal purification $(|\psi\rangle, A')$ is not dual to the entire original geometry: the purifying d.o.f. A' live on the enanglement wedge horizon. [Takayanagi, Umemoto; Nguyen,Devakul,Halbash,Zaletel,Swingle]



Reconstruction

- Consider the case in which B and C are contiguous. Then Σ is a geodesic exiting the wedge through the Rindler horizon

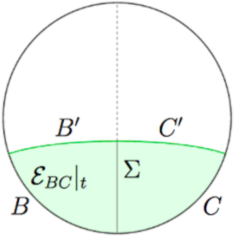


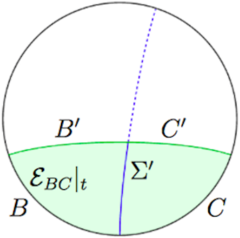
Reconstruction

- Consider the case in which B and C are contiguous. Then Σ is a geodesic exiting the wedge through the Rindler horizon
- For bulk reconstruction, we propose to use more general geodesics of this type: Σ'

$$P'(B:C|B') := \frac{\mathcal{A}(\Sigma')}{4G_N} = S_{BB'}\Big|_{((|\psi\rangle,A'))} .$$

Corresponds to an entanglement of purification. [RE,A. Güijosa,Pedraza]



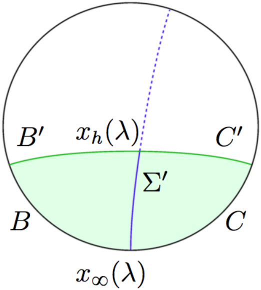


Differential Purification

- Using this slight generalization of the entanglement of purification, we can define the expression for differential purification

$$D := \oint d\lambda \left(\frac{\partial P'(x_\infty(\lambda), x_h(\bar{\lambda}))}{\partial \bar{\lambda}} \right)\Big|_{\bar{\lambda}=\lambda} .$$

- We show that it reproduces the length of spacelike curves: $D = A$.
[RE,A. Güijosa,Pedraza]

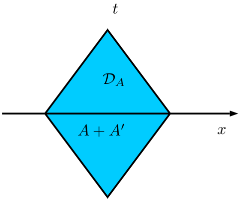


Conclusions

- Arbitrary spacelike curves can be reconstructed using entanglement entropy + entanglement of purification (and possibly null alignment).
- We can show the same is true within an arbitrary entanglement wedge in an arbitrary $aAdS_3$ bulk geometry.
- From the conceptual perspective, the most interesting aspect is that entanglement of purification seems to point to a radically new class of examples of holography, where the entanglement wedge is 'excised'.

Excised Duality

- In this configuration, all geodesics are avalable as ordinary entanglement entropy in AA'
- Natural notion of time evolution should be provided by (boundary & bulk) modular flow.
  [Jafferis et al; Faulkner, Lewkowicz; Wen;Freedman,Headrick]



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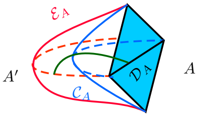

- Such idea deserves attention for future research.