Coarse grained quantum dynamics

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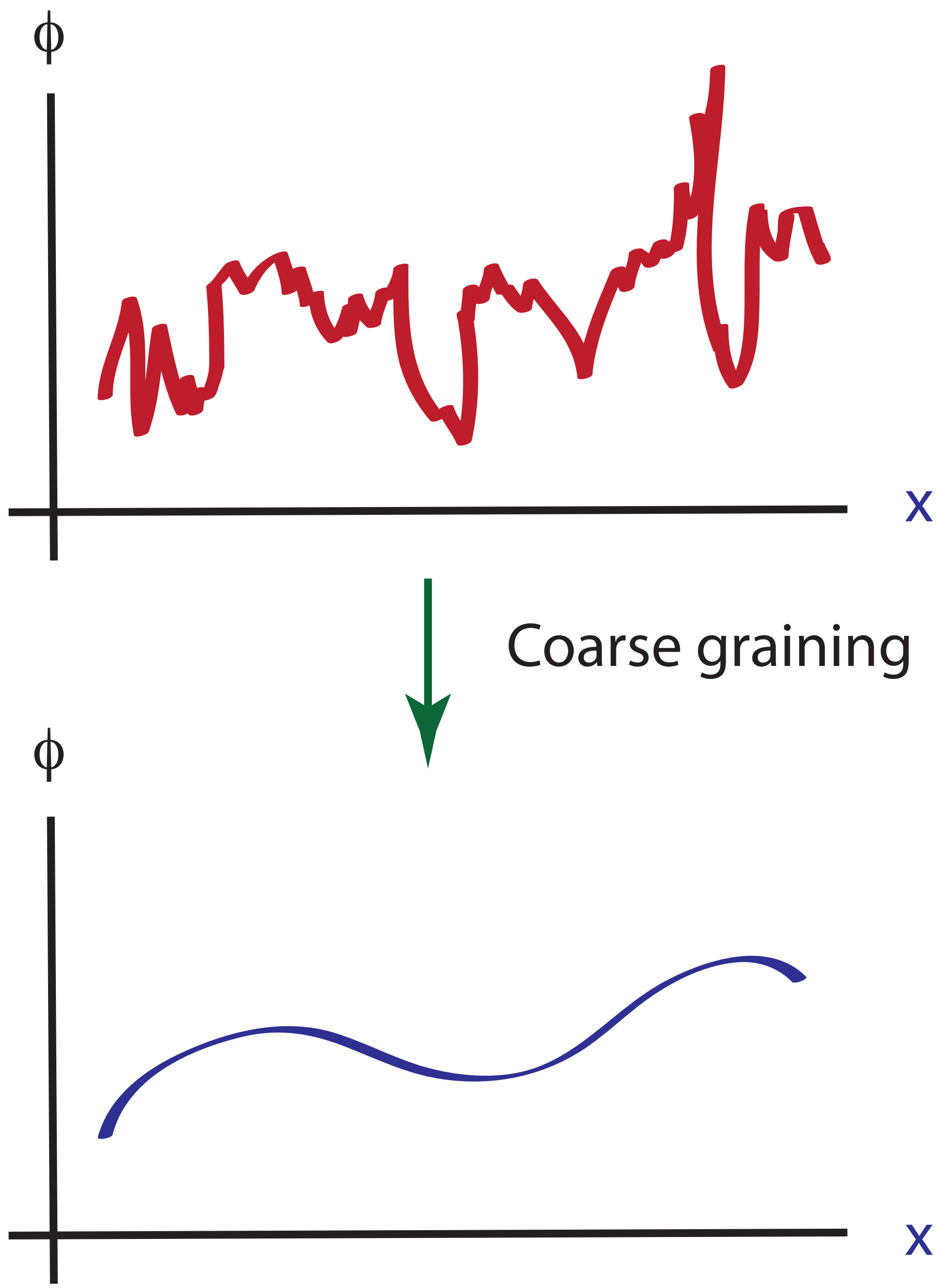
Based on:

- arxiv:1211.1729 with Vijay Balasubramanian and Monica Guica, JHEP 1301 (2013) 115.
- work in progress with Cesar Agon, VB, and Skyler Kasko

I. Introduction: coarse graining physical systems

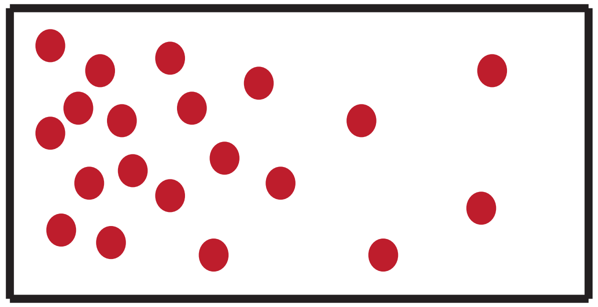
Typically our probes of a physical system (for example our eyes and ears) have finite resolution and accuracy

- Finite spatial resolution; access low momenta only
- Finite time resolution
- Probes have finite energy
- Noise, statistics, etc. limits accuracy



Coarse grained picture is simpler; requires fewer variables (for example, fewer Fourier modes)

Coarse grained picture reveals qualitative dynamics

1) 

Atoms in a box

$\longrightarrow$ $\rho(x, y; t)$ Fluid density

Complex atomic motion $\longrightarrow$

$$\frac{\partial}{\partial t}\rho(x, y; t) = D\vec{\nabla}\rho(x, y; t)$$

Diffusion equation

2) QED

$$\beta_{em} = \Lambda\frac{\partial}{\partial\Lambda}\alpha_{em}(\Lambda) > 0$$

Effective coupling weak at long distances:
charge screening

3) QCD

$$\beta_{st} = \Lambda\frac{\partial}{\partial\Lambda}\alpha_{st}(\Lambda) < 0$$

Effective coupling strong at long distances:
confinement, etc.

Given finite accuracy: can use "effective" theory with few parameters

d=3+1 scalar theory

$$[\phi] = (mass)$$

Coarse graining/"cutoff" mass scale Λ

$$S = \int d^4x \left[\overbrace{\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4}^{relevant} + \overbrace{\sum_{n=1}^{\infty} c_n \frac{\phi^{4+n}}{\Lambda^n} + \sum_{m=1}^{\infty} d_m \frac{(\partial\phi)^{2m+2}}{\Lambda^{4m}} + \ldots}^{irrelevant} \right]$$

Given process at energy E: $\sim \left(\frac{E}{\Lambda}\right)^n$ $\sim \left(\frac{E}{\Lambda}\right)^{4m}$

$E \ll \Lambda :$ can ignore most of the irrelevant operators at given level of accuracy

see Polchinski, hep-th/9210046

Goal: implement coarse graining in dynamical quantum setting

Scalar field in Schrödinger picture:

$$\phi(x) = \sum_{|k|\leq\Lambda} e^{ikx} a_{k,IR} + \sum_{|k|>\Lambda} e^{ikx} a_{k,UV} + \text{h.c.}$$

Split in Hilbert space: $\mathcal{H}_{IR} \times \mathcal{H}_{UV}$ ⟵ Generated by UV oscillators

Generated by IR oscillators

Local, interacting theory:

$$H = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4$$ ⟵ couples $\mathcal{H}_{IR},\ \mathcal{H}_{UV}$

$$\int dx \phi^4(x) = \int dp_1 \ldots dp_4 \delta(p_1 + \ldots p_4)\tilde{\phi}(p_1)\ldots\tilde{\phi}(p_4)$$

Interactions -> eigenstates of H entangled between $\mathcal{H}_{IR}$, $\mathcal{H}_{UV}$

Thomale, Arovas, Bernevig;
Balasubramanian, McDermott,
van Raamsdonk...

$$\lambda = 0: \quad |0\rangle = \prod_{k<\Lambda} |n_k = 0\rangle \prod_{k'>\Lambda} |n_{k'} = 0\rangle$$

$$\lambda > 0: \quad |0\rangle = |0\rangle_{\lambda=0} + \sum_{n_{k,IR}, n_{k',UV}} f_{n_{IR},n_{UV}}(\lambda) \prod_{k<\Lambda} |n_{k,IR}\rangle \prod_{k'>\Lambda} |n_{k',UV}\rangle$$

What do we want to calculate?

Assume observables built from $a_{k,IR}$ "long wavelength"

Consider ground state perturbed by IR operator

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}Ht}\mathcal{O}_{IR}|0\rangle$$

Probability of finding IR degrees of freedom in state $|a\rangle \in \mathcal{H}_{IR}$

$$\begin{aligned} P(a) &= \sum_{|u\rangle \in \mathcal{H}_{UV}} \left|\langle u|\langle a|e^{-\frac{i}{\hbar}Ht}\mathcal{O}_{IR}|0\rangle\right|^2 \\ &= \mathrm{tr}\mathbb{P}_a e^{-\frac{i}{\hbar}Ht}\mathcal{O}_{IR}|0\rangle\langle 0|\mathcal{O}_{IR}^\dagger e^{\frac{i}{\hbar}Ht} \\ &= \mathrm{tr}_{\mathcal{H}_{IR}}\mathbb{P}_a \rho_{IR}(t) \end{aligned}$$

$$\rho(t) = \mathrm{tr}_{\mathcal{H}_{UV}}|\psi(t)\rangle\langle\psi(t)|$$

Most important dynamical object

$$\mathbb{P}_a = |a\rangle\langle a|$$

- Typically a mixed state
- Dynamics: open quantum system

$$\rho(t) = \mathrm{tr}_{\mathcal{H}_{UV}} |\psi(t)\rangle\langle\psi(t)|$$

Entanglement -> generally a mixed state

$$|\psi\rangle = \sum_{n_{k,IR}, n_{k',UV}} g_{n_{IR}, n_{UV}}(\lambda) \prod_{k<\Lambda} |n_{k,IR}\rangle \prod_{k'>\Lambda} |n_{k',UV}\rangle$$

$$\downarrow$$

$$\rho = \sum_{n_{k,IR}, n_{k',UV}; m_{q,IR}} g^*_{m_{IR}, n_{UV}} g_{n_{IR}, n_{UV}} \prod_{k<\Lambda, q<\Lambda} |n_{k,IR}\rangle\langle m_{q,IR}|$$

$$\neq |\phi_{IR}\rangle\langle\phi_{IR}|$$

Interactions transfer energy, information between $\mathcal{H}_{IR}$, $\mathcal{H}_{UV}$

"Open quantum system"

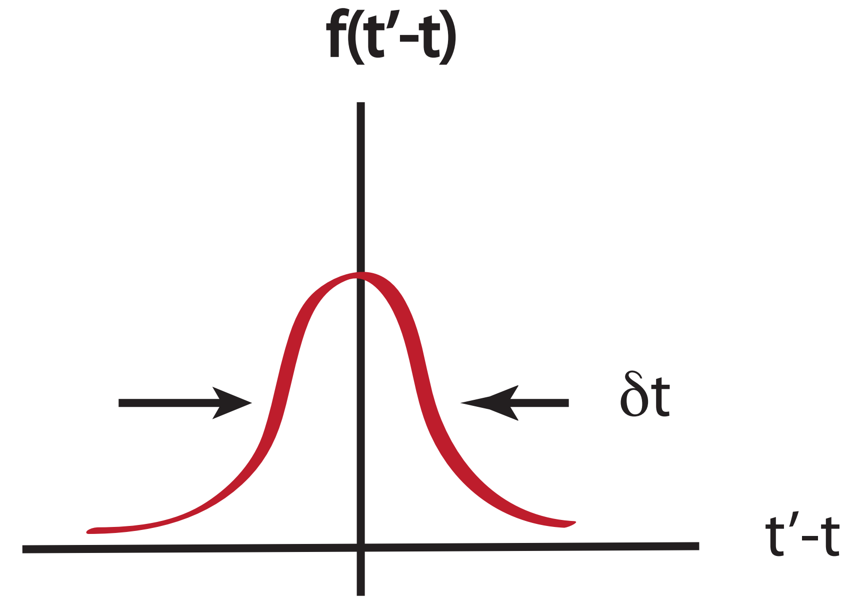
$$i\hbar \frac{\partial}{\partial t}\rho \neq [H, \rho]$$

Our main question:

Assume finite time resolution δt

$$\bar{\rho}(t) = \int dt' \, f_{\delta t}(t' - t)\rho(t)$$

$$i\hbar \, \partial_t \bar{\rho} = \, ?$$



- Given finite accuracy, can we parametrize LHS with a few operators?
- Can we organize LHS in powers of $\delta t, \Lambda^{-1}$?

Caveats

(1) We are not tracing out high *energies* $E \gg \Lambda \ , \delta t^{-1}$

$$M_{sun} c^2 \sim 10^{54} \ GeV$$

But solar physics well described by Standard Model cut off below 1 TeV

(2) Momentum space decomposition of $\mathcal{H}$ not universal

Headrick, AL, and Roberts
arxiv:1209.2428

1+1-d Bose-Fermi duality

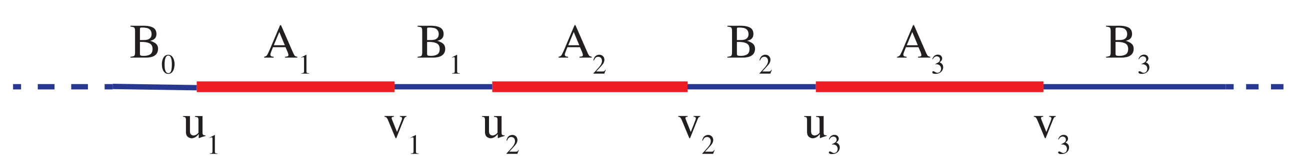
$$S = \int d^2x \frac{1}{2}(\partial\varphi)^2 \quad \Leftrightarrow \quad S = \int d^2x \left(\bar{\psi}\gamma\cdot\partial\psi - \lambda(R)(\bar{\psi}\gamma\psi)^2\right)$$

$$\varphi \equiv \varphi + 2\pi R$$

(with gauged $\mathbb{Z}_2$ fermion number)

Ground state: no momentum space entanglement

Ground state entangled in momentum space



$$\rho = \mathrm{tr}_{\cup_i B_i} |0\rangle\langle 0|$$

$$S_n = \frac{1}{1-n} \ln \mathrm{tr} \rho^n$$ duality invariant (spectrum and OPEs of local operators invariant!)

II. Wilsonian renormalization

Our formalism a variant of Wilson's approach(es)

Addresses different questions from those formalisms

A. Decimation of path integral

Wilson

$$\langle 0_{out}|0_{in}\rangle = \int D\phi_{UV} D\phi_{IR} e^{iS(\phi_{UV},\phi_{IR})}$$

$$= \int D\phi_{IR} e^{iS_{IR}(\phi_{IR})}$$

Wilsonian action

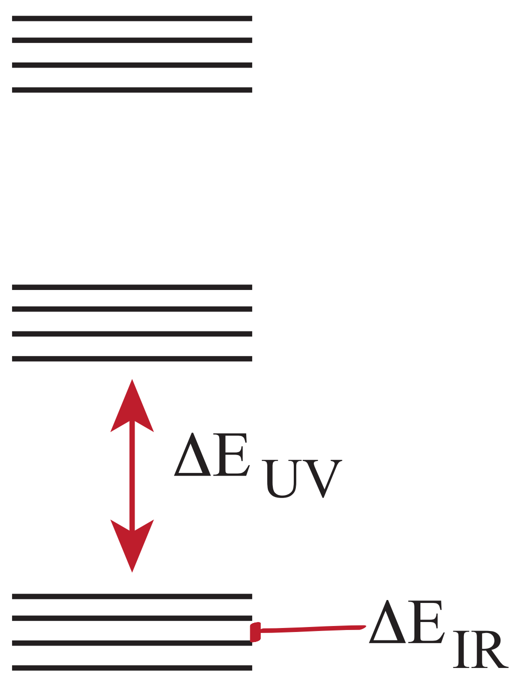
If $|0\rangle$ entangled between UV, IR, this integrating out at the level of amplitudes assumes knowledge of final state of UV.

Path integral for inclusive transition probabilities:
Wilsonian action -> Feynman-Vernon influence functional for ϕ_{IR}

B. Hamiltonian renormalization

Wilson;
Glazek and Wilson; see
also Peskin 1405.7086

Hierarchical structure of energy levels in QFT:



$$H = H_{IR} + H_{UV} + \lambda h_{UV,IR}$$

$$= \begin{pmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & H_{22} & H_{21} & H_{20} \\ \cdots & H_{12} & H_{11} & H_{10} \\ \cdots & H_{02} & H_{01} & H_{00} \end{pmatrix}$$

Bands at $\lambda = 0$

Diagonalize UV modes:

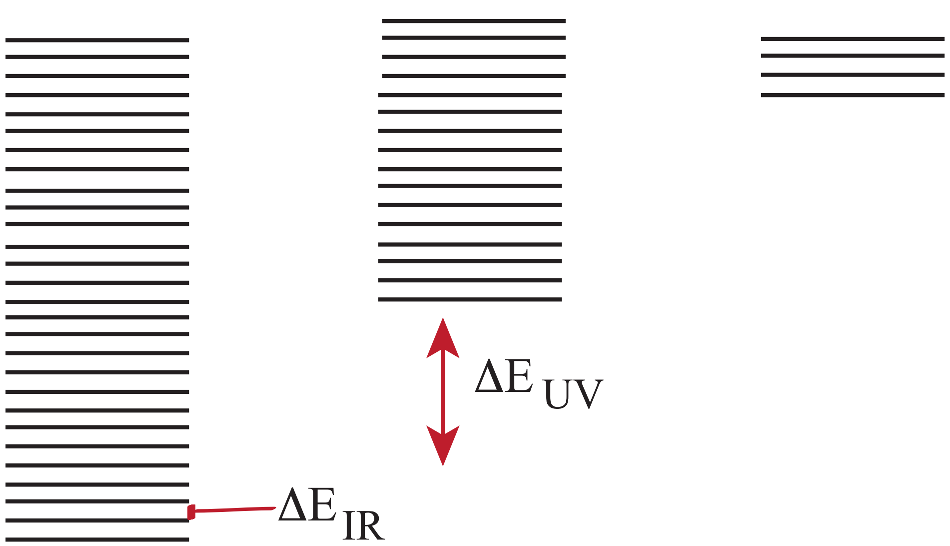
$$U^\dagger H U = \begin{pmatrix} \dots & \dots & \dots & 0 \\ \dots & H'_{22} & H'_{21} & 0 \\ \dots & H'_{12} & H'_{11} & 0 \\ 0 & 0 & 0 & H'_{00} \end{pmatrix}$$

Write H'_{00} in terms of renormalized IR variables: will have support on microscopic scales

For our purposes:

- What if our devices coarse grained wrt unrenormalized variables?
- Interested in high energy ($E \gg E_{UV}$) states made from low energy ($E \sim E_{IR}$) quanta. The above still works if indices label different towers of states with IR spacing

Alexanian and Moreno

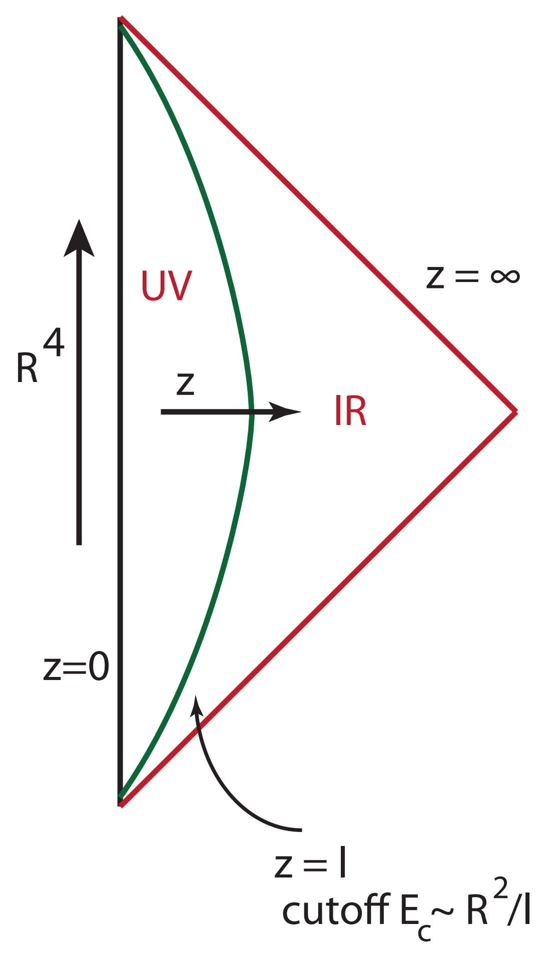


C. "Holographic" Wilsonian renormalization

(I) Holographic RG and equations of motion

Akhmedov; de Boer and Verlinde^2, Balasubramanian and Kraus; Porrati and Starinets; Klebanov and Strassler; Lawrence and Sever

Large N 4d CFT $\longleftrightarrow$ String theory on $AdS_5 \times M_5$



$$ds^2_{AdS} = \frac{R^2_{ads}}{z^2}\left(dz^2 + dx_4^2\right)$$

"Scale radius duality": $\Lambda \sim R^2_{ads}/z$

5d SUGRA fields $\phi_i(x,z) \leftrightarrow \mathcal{O}_i$ $\longleftarrow$ Single trace op.

$$z \to 0 \;:\; \phi \sim z^{d-\Delta_{\mathcal{O}}}\lambda(x) + z^{\Delta_{\mathcal{O}}}\langle\mathcal{O}\rangle$$

$$\Delta_{\mathcal{O}} = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m_\phi^2 R^2_{qds}}$$

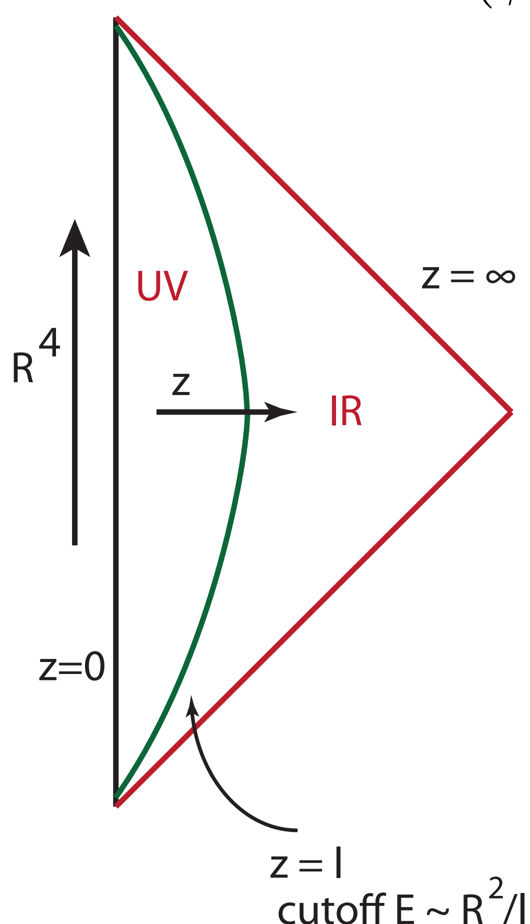
Witten; Gubser, Klebanov, and Polyakov; Balasubramanian, Kraus, and Lawrence; BKL and Trivedi

$$S = S_{CFT} + \delta S \;; \quad \delta S = \sum_i \int d^4x \lambda_i(x)\mathcal{O}_i(x)$$

"Hamiltonian" equations for radial evolution of ϕ $\longleftrightarrow$ RG equations for $\lambda(x)$

$S(\phi, z)$ generating function for $\langle\psi|\mathcal{O}_1 \ldots \mathcal{O}_n|\psi\rangle$

de Boer and Verlinde^2; Lawrence and Sever



Satisfies bulk Hamilton-Jacobi equations

$$\partial_z S(\phi; z) + H\left(\pi_\phi = \frac{\delta S}{\delta\phi}, \phi\right) = 0$$

Dual to Callan-Symanzik equations for perturbed CFT

$$\left(\Lambda\frac{\partial}{\partial\Lambda} + \sum_i \beta_i(\lambda)\frac{\delta}{\delta\lambda_i}\langle\psi|\mathcal{O}_1 \ldots \mathcal{O}_n|\psi\rangle\right) - \sum_{k=1}^{n} \gamma_k^{i_k}\langle\ldots\mathcal{O}_{i_k}\ldots\rangle = 0$$

Hamilton's equations $\qquad \dfrac{d\phi}{dz} = \dfrac{\delta H(\pi,\phi)}{\delta\pi}\Big|_{\pi=\frac{\delta S}{\delta\phi}}$

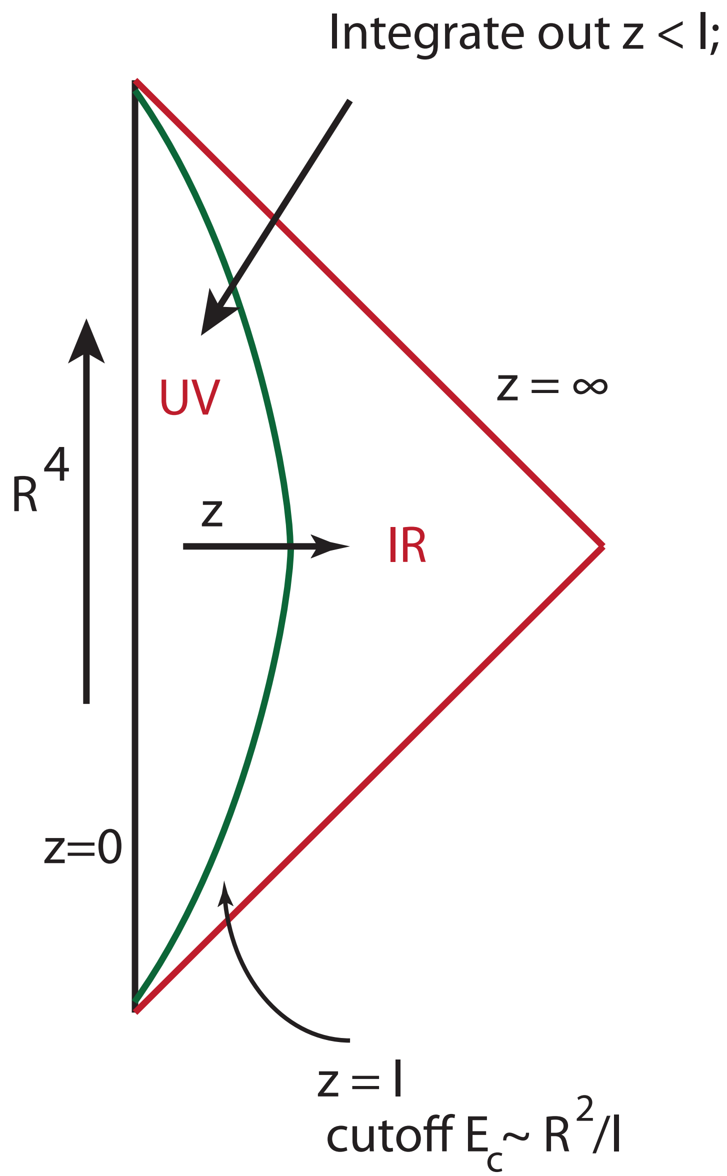
Dual to RG flow equations $\qquad \Lambda\dfrac{d}{d\Lambda}\lambda_i(\Lambda) = \beta_i(\lambda)$

Conceptual issues

- Semiclassical formulation: how to take into account quantum fluctuations?
- Non-Wilsonian: how to describe "integrating out" degrees of freedom?
- Formalism ignored "double trace" operators that appear at leading order in 1/N

(2) Holographic Wilsonian renormalization

Heemsekerk and Polchinski;
Faulkner, Liu, and Rangamani;
Balasubramanian, Guica, and AL



$$Z_{IR}(\lambda) = \int_{\phi(x,l)=\lambda(x)} D_{z>l}\ \phi e^{iS(\phi)}$$
$$= \langle e^{-\int d^4x \lambda(x)\mathcal{O}(x)}\rangle_{CFT,\Lambda}$$

$$Z_{UV}(\lambda) = \int_{\phi(x,l)=\lambda(x)} D_{z<l}\ \phi e^{iS(\phi)} = e^{i\mathcal{S}(\lambda)}$$

$$Z_{bulk} = \int d\lambda Z_{UV}(\lambda) Z_{IR}(\lambda)$$
$$= \langle e^{i\int d^4x g_i(x;\Lambda)\mathcal{O}_i + i\int d^4x d^4y \gamma_{ij}(\Lambda;x-y)\mathcal{O}_i(x)\mathcal{O}_j(y)+\dots}\rangle_\Lambda$$

g, γ : running couplings: no direct relationship to $\phi(x;z)$

Choice of scheme

$$\Lambda \to \infty : \langle \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(0)\rangle \to_{|x|\to 0} \frac{1}{|x|^{2\Delta}} \quad \longleftrightarrow \quad \langle \tilde{\mathcal{O}}(q)\tilde{\mathcal{O}}(q')\rangle \sim \delta(q+q')q^{2\Delta-d}$$

prescription above implies for finite Λ

$$\frac{\delta}{\delta\tilde{\lambda}(q)}\frac{\delta}{\delta\tilde{\lambda}(q')}Z_{IR} = \langle \tilde{\mathcal{O}}(q)\tilde{\mathcal{O}}(q')\rangle_{\Lambda,CFT}$$

$$= \delta(q+q')\left(f_0(q) + q^{2\Delta-d}f_1(q) + q^{2(2\Delta-d)}f_2(q) + \dots\right)$$

$f_k(q)$ analytic in q

Balasubramanian, Guica, and AL

$$\Rightarrow \Lambda < \infty \ : \ \lambda_{i,\Lambda}(x) \text{ couples to } \mathbf{1}, \partial^k\mathcal{O}, \dots$$

Change of scheme

Dong et al; BGL

$$Z_{IR}(\phi_\Lambda) = e^{iS^{\Lambda}_{ct}(\phi_\Lambda)}\, Z^{ren}_{IR}(\phi_\Lambda) = e^{iS^{\Lambda}_{ct}(\phi_\Lambda)} \langle e^{iN^2 \int d^dx\, \Lambda^{d-\Delta}\phi_\Lambda \mathcal{O}} \rangle_{ren,\Lambda}$$

$$Z_{UV} = e^{-iS^{\Lambda}_{ct}} Z^{ren}_{UV}$$

Analytic in derivatives

$$Z_{bulk} = \int d\lambda Z^{ren}_{UV}(\lambda) Z^{ren}_{IR}(\lambda)$$
$$= \langle e^{i\int d^4x g^{ren}_i(x;\Lambda)\mathcal{O}_i + i\int d^4x d^4y \gamma^{ren}_{ij}(\Lambda; x-y)\mathcal{O}_i(x)\mathcal{O}_j(y) + \ldots} \rangle_{ren,\Lambda}$$

For sufficiently small q, can choose scheme such that

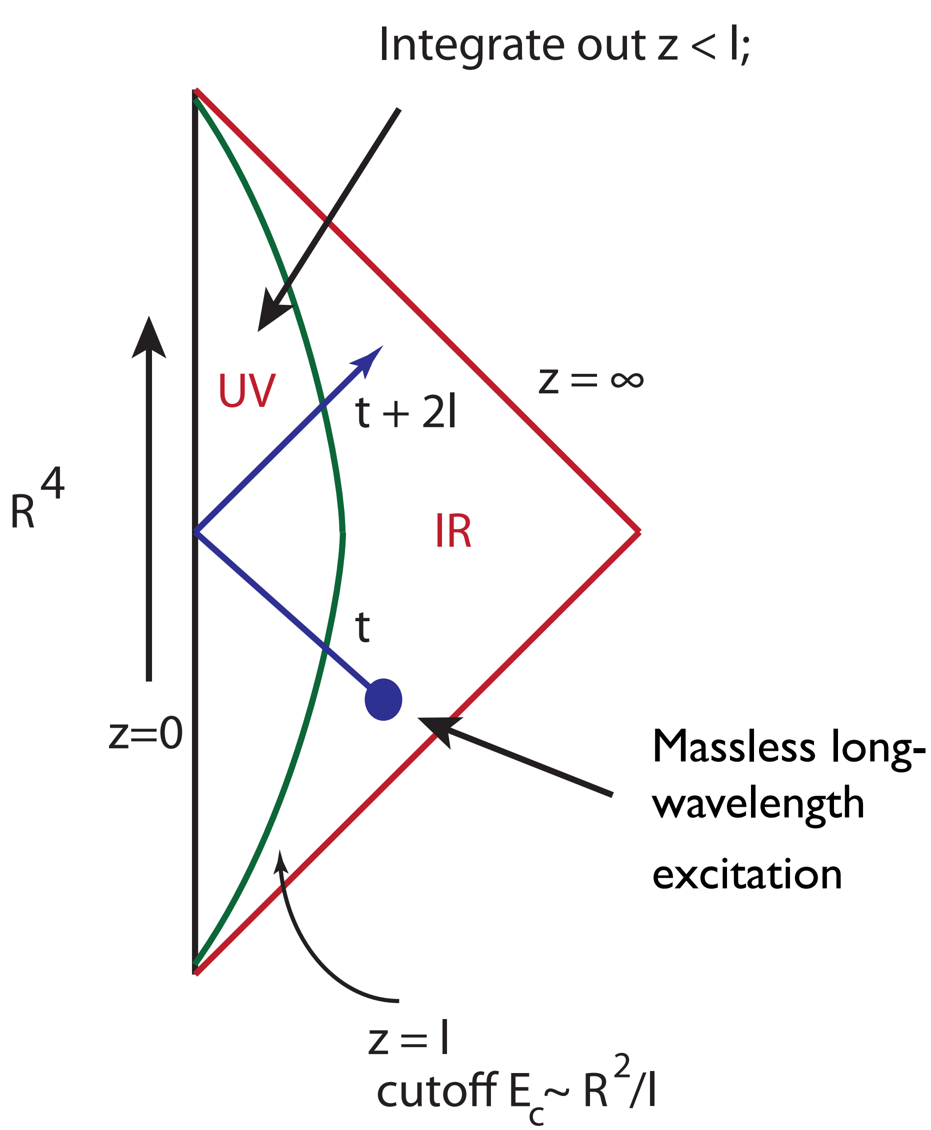
- $\langle \tilde{\mathcal{O}}(q)\tilde{\mathcal{O}}(q')\rangle_{\Lambda,CFT} \sim \delta(q+q')\left(q^{2\Delta-d}f_1(q) + q^{2(2\Delta-d)}f_2(q) + \ldots\right)$
- $g^{ren}(x, \Lambda = R^2_{ads}/z)$ solves spacetime equations of motion in x,z

Meaning of double-trace operators

$$S = S_{CFT} + \int d^4x g_i(\Lambda; x)\mathcal{O}_i + \int d^4x d^4y \mathcal{O}(x)\mathcal{O}(y)\gamma(x, y; \Lambda) + \dots$$

nonlocal at scales

$$\delta x, \delta t \sim \Lambda^{-1}$$



γ induced even if UV theory is unperturbed CFT

describes transfer of excitations from IR $\leftrightarrow$ UV

Fits framework of introduction: "IR" z < 1/Ec is open quantum system

III. Open quantum dynamics for $\mathcal{H}_{IR}$

Hilbert space: $\mathcal{H} = \mathcal{H}_{IR} \times \mathcal{H}_{UV}$

Observable "long wavelength" quanta

Hamiltonian: $H = H_{IR} + H_{UV} + \lambda V_{IR,UV}$

Characteristic energy: ΔE_{IR} ΔE_{UV}

Reduced density matrix: $\rho(t) = \mathrm{tr}_{\mathcal{H}_{UV}} |\Psi(t)\rangle\langle\Psi(t)|$

What are dynamics of $\rho(t)$?

Time averaging

Expect finite spatial and temporal resolution $\delta t = 1/E_c \gtrsim 1/E_{UV}$

Describe via "window function" $f_{E_c}(t, t')$

Peak of function

e.g. $f_{E_c}(t,t') = \frac{E_c}{\sqrt{\pi}} e^{-(t-t')^2 E_c^2}$

Given operator A(t), $\overline{A(t)} \equiv \int dt' f_{E_c}(t,t') A(t')$

$$\overline{A(t)B(t)} = \sum_{n=0}^{\infty} \frac{1}{(2E_c^2)^n n!} d_t^n \overline{A(t)} d_t^n \overline{B(t)} + \dots$$

- A(t) has time dependence at scale $1/E_{UV} \ll 1/E_c \; ; \; \Rightarrow \overline{A(t)} \sim \mathcal{O}\left(e^{-E_{UV}^2/E_c^2}\right)$
- A, B have time dependence at scale

$$1/E_{IR} \gg 1/E_c \; ; \; \overline{A(t)B(t)} \sim \overline{A(t)}\;\overline{B(t)} + \mathcal{O}\left(\frac{E_{IR}^2}{E_c^2}\right)$$

Wish to compute RHS of $i\hbar \partial_t \overline{\rho(t)} = \, ?$ in terms of time averaged operators

Initial state

At present we only have results for $|\Psi(0)\rangle = |\Psi_{IR}\rangle|\Psi_{UV}\rangle$

Not most general but it appears naturally in this case:

$$\mathcal{H}_{IR} = \mathbb{C}^{2j_{IR}} \ , \ \mathcal{H}_{UV} = \mathbb{C}^{2j_{UV}}$$

$$H = -\mu_{IR} B \hat{S}^z_{IR} - \mu_{UV} B \hat{S}^z_{UV} + \epsilon \vec{S}_{IR} \cdot \vec{S}_{UV} \ ; \ \mu_{IR} \ll \mu_{UV}$$

Ground state is independent of ϵ: $|0\rangle = |j_{IR}\rangle|j_{UV}\rangle$

Consider states of the form: $|\Psi\rangle = \left(\hat{S}^-_{IR}\right)^k |j_{IR}\rangle|j_{UV}\rangle$

$\rho(t), \overline{\rho(t)}$ will become mixed for t > 0

Perturbation theory

$$|\Psi(t)\rangle = |\Psi^{(0)}(t)\rangle + \lambda|\Psi^{(1)}(t)\rangle + \lambda^2|\Psi^{(2)}(t)\rangle + \dots$$

$$\rho(t) = \mathrm{tr}_{\mathcal{H}_{UV}}|\Psi(t)\rangle\langle\Psi(t)| = \rho^{(0)}(t) + \lambda\rho^{(1)}(t) + \lambda^2\rho^{(2)}(t) + \dots$$

$$i\hbar\partial_t\rho(t) = \mathrm{tr}_{\mathcal{H}_{UV}}[H, |\Psi(t)\rangle\langle\Psi(t)|] = [H_{eff}, \rho] + \Gamma(t, \rho(0))$$

$$i\hbar\partial_t\bar{\rho}(t) = [\bar{H}_{eff}, \rho] + \bar{\Gamma}(t, \rho(0))$$

$$\bar{H}_{eff} = H_{IR} + \lambda\bar{H}_{eff}^{(1)} + \lambda^2\bar{H}_{eff}^{(2)} + \dots$$

$$\bar{\Gamma} = \lambda\bar{\Gamma}^{(1)} + \lambda^2\bar{\Gamma}^{(2)} + \dots$$

2nd order perturbation theory

$$i\hbar\partial_t\overline{\rho}(t) = [\overline{H}_{eff}, \overline{\rho}] + \overline{\Gamma}(t; \{\rho(0)\})$$

Consider basis $|u\rangle$ of $\mathcal{H}_{UV}$; $|\Psi_{UV}\rangle = |\bar{u}\rangle$

$$\overline{H}_{eff} = H_{IR} + \lambda\langle\bar{u}|V|\bar{u}\rangle - \lambda^2\left(\sum_{u\neq\bar{u}}\frac{V_u^\dagger V_u}{E_u - E_{\bar{u}}} + \mathcal{O}\left(\frac{\Delta E_{IR}}{\Delta E_{UV}}\right)\right)$$

$$\mathcal{O}\left(\frac{\Delta E_{IR}}{\Delta E_{UV}}\right)$$

$$\overline{\Gamma}(t) = \{\overline{A}^{(2)}(t), \overline{\rho}^{(0)}(t)\} + \lambda^2\left(\sum_{u\neq\bar{u}}\frac{[V_u, H_{IR}]\rho^{(0)}(t)V_u^\dagger + V_u\rho^{(0)}(t)[V_u^\dagger, H_{IR}]}{(E_u - E_{\bar{u}})^2} + \mathcal{O}\left(\frac{\Delta E_{IR}^2}{\Delta E_{UV}^2}\right)\right)$$

$$\overline{A}^{(2)}(t) = -\frac{\lambda^2}{2}\left(\sum_{u\neq\bar{u}}\frac{[V_u^\dagger V_u, H_{IR}]}{(E_u - E_{\bar{u}})^2} + \mathcal{O}\left(\frac{\Delta E_{IR}^2}{\Delta E_{UV}^2}\right)\right) \quad ; \quad V_u = \langle u|V|\bar{u}\rangle$$

- Double power series in λ , $\frac{\Delta E_{IR}}{\Delta E_{UV}}$ -- related to Born-Oppenheimer approx
- $\overline{\Gamma}(t)$ parametrizes non-unitary evolution
- $\overline{\Gamma}(t)$ due to transitions in $\mathcal{H}_{UV}$; occurs at higher order in $\lambda, \Delta E_{IR}/\Delta E_{UV}$
- $\overline{\Gamma}(t)$ has time dependence at IR scale through $\rho^{(0)}(t) = e^{-iH_{IR}t}|\psi_{IR}\rangle\langle\psi_{IR}|e^{iH_{IR}t}$

UV-IR entanglement

$$S_n(t) = -\frac{1}{1-n}\operatorname{tr}\ln\rho^n(t)$$

$$\frac{dS_n(t)}{dt} = -\frac{n\operatorname{tr}\left(\rho^{n-1}\Gamma\right)}{i\hbar(1-n)\operatorname{tr}\rho^n(t)} \sim \frac{n}{i\hbar(n-1)}\operatorname{tr}\left(\rho^{(0)}\Gamma\right) + \mathcal{O}(\lambda^3)$$

Expectations from Born-Oppenheimer

Consider $\mathcal{H}_{IR} = L^2(\mathbb{R})$

$$V_{IR,UV} = V\left(x_{IR}, \{\mathcal{O}_{i,UV}\}\right)$$

$$\left(H_{UV} + \lambda V(x)\right)|u;x\rangle = E_u(x)|u;x\rangle$$

$$|\Psi(t)\rangle = \int dx \psi_{\bar{u}}(x,t)|x\rangle|\bar{u};x\rangle + \sum_{u\neq\bar{u}} \int dx \psi_u(x,t)|x\rangle|u;x\rangle$$

Corrections to Born-Oppenheimer; higher order in $\lambda, \Delta E_{IR}/\Delta E_{UV}$

Leading order in Born-Oppenheimer approximation

$$\rho = \mathrm{tr}_{\mathcal{H}_{UV}} |\Psi(t)\rangle\langle\Psi(t)|$$
$$= \int dx dy \psi_{\bar{u}}(x,t)\psi^*_{\bar{u}}(y,t) K_{\bar{u}}(x,y)|x\rangle\langle y|$$

$$K_{\bar{u}}(x,y) = \mathrm{tr}_{\mathcal{H}_{UV}} |\bar{u};x\rangle\langle\bar{u};y| = 1 + \lambda f(\lambda;x,y)$$

$$i\hbar\partial_t\rho = [H_{eff},\rho];\ \ H_{eff} = H_{IR} + E_{\bar{u}}(x)$$

Corrections: $i\hbar\partial_t\rho = [H_{eff},\rho] + \Gamma[\rho(0)]$

Occurs at higher order in Born-Oppenheimer

Consistent with our result $\Gamma \sim \mathcal{O}\left(\Delta E_{IR}/\Delta E_{UV}\right)$

Path integral approach

x = IR coordinates; X = UV coordinates

$$S[x,X] = S_{IR}(x) + S_{UV}(X) + S_{int}(x,X)$$

$$S_{int}(x,X) = \sum_i \int dt' \; \lambda A^{(x)}_{i,IR}(t')\mathcal{O}^{(X)}_{i,UV}(t')$$

$$\rho(0) = \rho_{IR}(0)\rho_{UV}(0)$$

$$\rho(t) = e^{-iHt}\rho(0)e^{iHt}$$

$$\rho_{IR}(t) = \int dX \langle X|e^{-iHt}\rho(0)e^{iHt}|X\rangle$$

$$\rho_{IR}(x,y;t) = \int dX \langle x|\langle X|e^{-iHt}\rho(0)e^{iHt}|X\rangle|y\rangle$$

Propagate backwards in time from y,X to initial state

Propagate forward in time from initial state to x, X

$$\rho_{IR}(x,y;t) = \langle x|\rho_{IR}(t)|y\rangle$$
$$= \int dx'dy'\mathcal{K}(x,y,t;x',y',0)\rho_{IR}(x';y';t=0)$$

Paths propagate backwards in time

$$\mathcal{K}(x,y,t;x',y',0) = \int D\tilde{x}D\tilde{y}e^{iS_{IR}[\tilde{x}]-iS_{IR}(\tilde{y})}\mathcal{F}(\tilde{x},\tilde{y})\Big|^{\tilde{x}(t)=x;\tilde{y}(t)=y}_{\tilde{x}(0)=x',\tilde{y}(0)=y'}$$

Feynman and Vernon;
Caldeira and Leggett

Paths propagate forward in time

"Influence functional"; result of integrating out X

Correct real-time analog of Wilsonian action

Contains same information as H_{eff}, Γ

$$\mathcal{F}(\tilde{x},\tilde{y}) = \int dR'dQ'dR\rho_{UV}(R',Q';0)$$
$$\times \int DRDQe^{iS_{UV}(R)-iS_{UV}(Q)+iS_{int}(\tilde{x},R)-iS_{int}(\tilde{y},Q)}\Big|^{R(t)=Q(t)=R}_{R(0)=R';Q(0)=Q'}$$

Compute $i\partial_t\rho_{IR}(t)$ in this framework: deduce relationship between $\mathcal{F}$ and H_{eff}, Γ

Perturbation theory $\mathcal{F} = 1 + \lambda \mathcal{F}^{(1)} + \lambda^2 \mathcal{F}^{(2)} + \ldots$

Remember: $S_{int}(x, X) = \sum_i \int dt' \; \lambda A^{(x)}_{i,IR}(t') \mathcal{O}^{(X)}_{i,UV}(t')$

$$\mathcal{F}^{(1)}(\tilde{x}, \tilde{y}, t) = i \int_0^t dt' \langle \mathcal{O}^{(X)}_a(t') \rangle_{UV,0} \left[A^{(\tilde{x})}_a(t') - A^{(\tilde{y})}_a(t') \right]$$

$$\mathrm{tr}\left[\rho^{(0)}_{UV}(0) \mathcal{O}_a(t') \right]$$

$$H^{(1)}_{eff} = \langle \mathcal{O}_a(t) \rangle_{UV} A_a(t)$$

Consistent with operator-based computation

$$
\begin{aligned}
\mathcal{F}^{(2)}(\tilde{x},\tilde{y};t) = &- \frac{1}{2}\int_0^t dt' dt'' G^F_{ab}(t',t'') A^{(\tilde{x})}_a(t') A^{(\tilde{x})}_b(t'') \\
&- \frac{1}{2}\int_0^t dt' dt'' \tilde{G}^F_{ab}(t',t'') A^{(\tilde{y})}_a(t') A^{(\tilde{y})}_b(t'') \\
&+ \int_0^t dt' dt'' G^W_{ab}(t',t'') A^{(\tilde{y})}_a(t') A^{(\tilde{x})}_b(t'')
\end{aligned}
$$

$$
\begin{aligned}
G^W_{ab}(t',t'') &= \mathrm{tr}_{UV}\rho_{UV}(0)\mathcal{O}_a(t')\mathcal{O}_b(t'') \\
G^F_{ab}(t',t'') &= \mathrm{tr}_{UV}\rho_{UV}(0)T\left[\mathcal{O}_a(t')\mathcal{O}_b(t'')\right] \\
\tilde{G}^F_{ab}(t',t'') &= \mathrm{tr}_{UV}\rho_{UV}(0)\tilde{T}\left[\mathcal{O}_a(t')\mathcal{O}_b(t'')\right] = G^F_{ab}(t',t'')^*
\end{aligned}
$$

Time anti-ordering

$$i\partial_t \rho = [H_{eff}, \rho] + \Gamma$$
$$\Gamma = \{A, \rho\} + \gamma$$

$$H^{(2)}_{eff} = \frac{1}{2}\int_0^t dt' \ \mathrm{Im}\ G^F_{ab}(t,t')A_a(t)A_b(t)$$
$$A^{(2)} = \frac{1}{2}\int_0^t dt' \ \mathrm{Re}\ G^F_{ab}(t,t')A_a(t)A_b(t)$$

- Work in progress
- Need to understand time averaging better in this framework

IV. Conclusions

A. Summary

$$\mathcal{H} = \mathcal{H}_{IR} \times \mathcal{H}_{UV}$$

$$H = H_{IR} + H_{UV} + \lambda V_{IR,UV}$$

$$\rho(t) = \mathrm{tr}_{\mathcal{H}_{UV}} |\Psi(t)\rangle\langle\Psi(t)|$$

$$i\hbar\partial_t \overline{\rho}(t) = [\overline{H}_{eff}, \overline{\rho}] + \overline{\Gamma}(t; \{\rho(0)\})$$

- Parametrizes non-unitary evolution of open system
- Local in time only after time averaging
- Appears only at $\mathcal{O}\left(\Delta E_{IR}/\Delta E_{UV}\right)$ (correction to Born-Oppenheimer), $\mathcal{O}\left(\lambda^2\right)$

B. Additional questions

Some natural questions:

- Wilsonian EFT-like organization of $\overline{H}_{eff}, \overline{\Gamma}$ in power series in $\frac{1}{E_{UV}}$, $\frac{1}{E_c}$
- Efficient computational scheme
- RG equations for ρ, H_{eff}, Γ
- Holographic interpretation? (Note importance of time averaging)
- Formulate for strongly interacting DOF (no quasiparticles E(k))
- What would we have to do to H to spoil decoupling of UV, IR
- What systems lead to excitations spending long time in UV?