

Aspects of Entanglement: Entropies, Negativity and Causal Holographic Information



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- A. Coser, L. Tagliacozzo and E.T.; [1309.2189] (JSTAT, special issue)
P. Calabrese, L. Tagliacozzo and E.T.; [1302.1113] (JSTAT)
P. Calabrese, J. Cardy and E.T.; [1210.5359] (JSTAT)
[1206.3092] (PRL)
- V. Hubeny, H. Maxfield, M. Rangamani and E.T.; [1306.4004] (JHEP)
V. Hubeny, M. Rangamani and E.T.; [1306.4324] (JHEP)
[1302.0853] (JHEP)

MexStrings 2014
Colima, June 2014

Plan of the talk

Quantum Field Theory

CFT

Renormalization Group

AdS/CFT

Black Holes physics

Condensed Matter Theory

Lattice models

Entanglement

Quantum Information Theory

JPA special issue vol. 42 (2009) eds.: P. Calabrese, J. Cardy and B. Doyon



Entanglement in 2D CFT:

- Entanglement entropies for disjoint intervals
- Entanglement for mixed states: Negativity



Entanglement in AdS/CFT:

- Causal Holographic Information

Entanglement entropy: definition

- Quantum system ($\mathcal{H}$) in the ground state $|\Psi\rangle$
Density matrix $\rho = |\Psi\rangle\langle\Psi| \implies \text{Tr}\rho^n = 1$

- Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

- A 's reduced density matrix $\rho_A = \text{Tr}_B \rho$

if ρ describes a pure state then ρ_A describes a mixed state

- Entanglement entropy $\equiv$ Von Neumann entropy of ρ_A

$$S_A = -\text{Tr}_A(\rho_A \log \rho_A)$$

- Replica trick

$$S_A = \lim_{n \rightarrow 1} \frac{\log(\text{Tr} \rho_A^n)}{1-n} = - \lim_{n \rightarrow 1} \frac{\partial}{\partial n} \text{Tr} \rho_A^n$$

Rényi entropies



Entanglement entropy: some properties

■ Bipartition $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ $\rho = |\Psi\rangle\langle\Psi|$ pure state

$$|\Psi\rangle = \sum_k c_k |\Psi_k\rangle_A |\Psi_k\rangle_B$$
$$c_k \geq 0 \quad \sum_k c_k^2 = 1$$

Schmidt decomposition

$$S_A = - \sum_k c_k^2 \log c_k^2 = S_B$$

$\Rightarrow S_A$ is not extensive

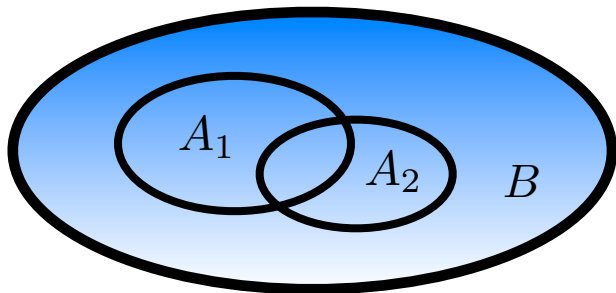
Araki-Lieb inequality

■ ρ mixed state $\delta S_A \equiv S_A - S_B$

$$|\delta S_A| \leq S_{A \cup B} = S_\rho$$

■ Subadditivity $A_1 \cap A_2 = \emptyset$ $S_{A_1} + S_{A_2} \geq S_{A_1 \cup A_2}$

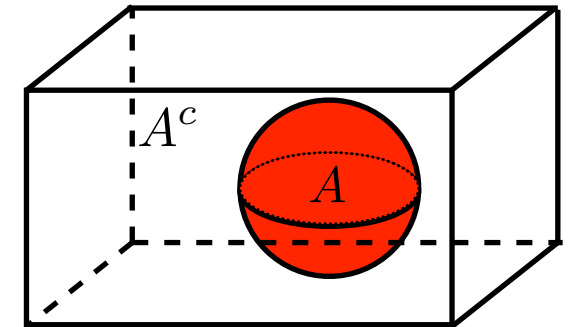
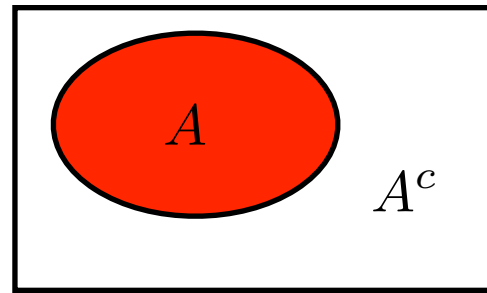
■ Strong Subadditivity



$$S_{A_1} + S_{A_2} \geq S_{A_1 \cup A_2} + S_{A_1 \cap A_2}$$
$$S_{A_1} + S_{A_2} \geq S_{A_1 \setminus A_2} + S_{A_2 \setminus A_1}$$

Geometric entropy: area law

- Assume that A and B correspond to a spatial bipartition of the system



- Area law: In d spatial dimensions when $\rho = |\Psi\rangle\langle\Psi|$ ($S_A = S_{A^c}$)

$$S_A \propto \frac{\text{Area}(\partial A)}{a^{d-1}} + \dots$$

[Bombelli, Koul, Lee, Sorkin, PRD (1986)]

[Srednicki, PRL (1993)]

- ➔ In $1 + 1$ CFTs at $T = 0$
[Holzhey, Larsen, Wilczek, NPB (1994)]
[Calabrese, Cardy, JSTAT (2004)]

$$S_A = \frac{c}{3} \log \frac{\ell}{a} + \text{const}$$

- ➔ In $2 + 1$ CFTs for a circle
[Myers, Sinha, JHEP (2011)]
[Klebanov, Pufu, Safdi, JHEP (2011)]
[Casini, Huerta, PRD (2012)]

$$S_A = \gamma \frac{2\pi R}{a} - f$$

Replica trick for the entanglement entropy

$$S_A = -\text{Tr}(\rho_A \log \rho_A) = \lim_{n \rightarrow 1} \frac{\log(\text{Tr} \rho_A^n)}{1-n} = - \lim_{n \rightarrow 1} \frac{\partial}{\partial n} \text{Tr} \rho_A^n$$

$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

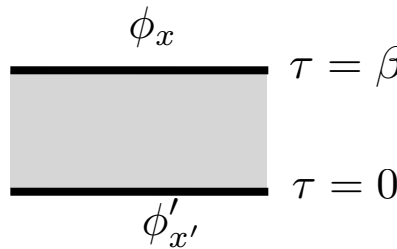
Rényi entropies

[Callan, Wilczek, PLB (1994)]

[Holzhey, Larsen, Wilczek, NPB (1994)]

[Calabrese, Cardy, JSTAT (2004)]

Example: density matrix ρ in a thermal state at temperature $T = 1/\beta$



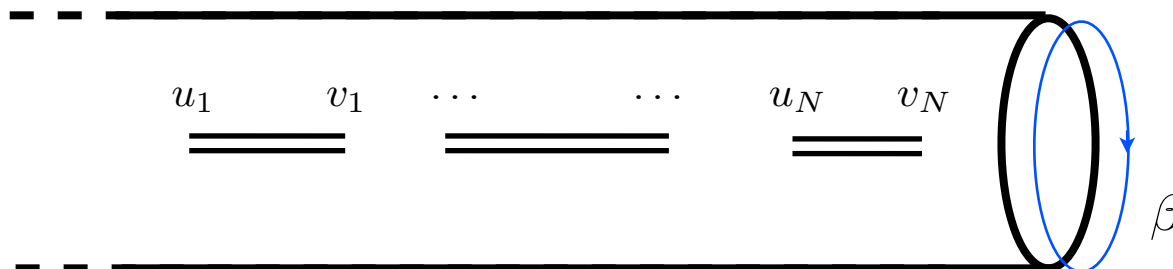
$$\rho(\{\phi_x\}|\{\phi'_{x'}\}) = \frac{1}{\mathcal{Z}} \int [d\phi(y, \tau)] \prod_{x'} \delta(\phi(y, 0) - \phi'_{x'}) \prod_x \delta(\phi(y, \beta) - \phi_x) e^{-S_E}$$

$$\mathcal{Z} = \text{Tr} e^{-\beta H}$$

The trace sews together the edges at $\tau = 0$ and $\tau = \beta$ providing a cylinder with circumference of length β .

$$\rho_A = \text{Tr}_B \rho$$

$$A = (u_1, v_1) \cup \dots \cup (u_N, v_N)$$



The trace Tr_B sews together only the points $\notin A$.

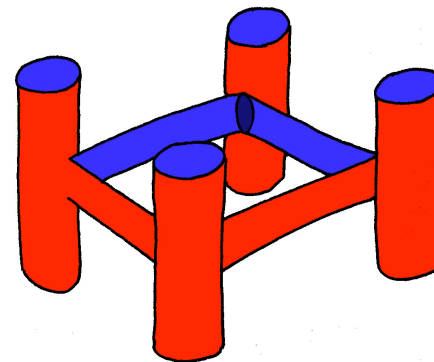
Open cuts are left along the disjoint intervals (u_j, v_j) .

Entanglement entropies & Riemann surfaces

- n copies sewed together cyclically along the cuts

$$\text{Tr} \rho_A^n = \frac{\mathcal{Z}_{N,n}}{\mathcal{Z}^n}$$

$$“\rho_A^{ij} \rho_A^{jk} \rho_A^{kl} \rho_A^{li}” =$$



- Partition function on the n sheeted Riemann surface $\mathcal{R}_{N,n}$

$$\mathcal{Z}_{N,n} = \int_{\mathcal{C}_A} [d\varphi_1 \cdots d\varphi_n]_{\mathbf{C}} \exp \left[- \int_{\mathbf{C}} dz d\bar{z} (\mathcal{L}[\varphi_1](z, \bar{z}) + \cdots + \mathcal{L}[\varphi_n](z, \bar{z})) \right]$$

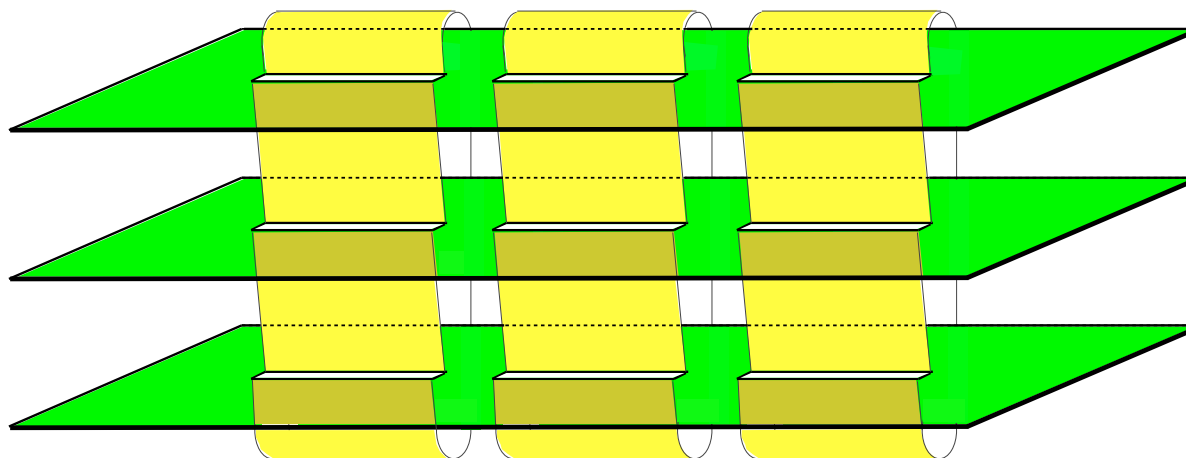
$$\mathcal{C}_A : \quad \varphi_i(x, 0^+) = \varphi_{i+1}(x, 0^-) \quad x \in A = \cup_{j=1}^N A_j \quad i = 1, \dots, n$$

N cuts
 n sheets



$$g = (N - 1)(n - 1)$$

$\mathcal{R}_{3,3}$



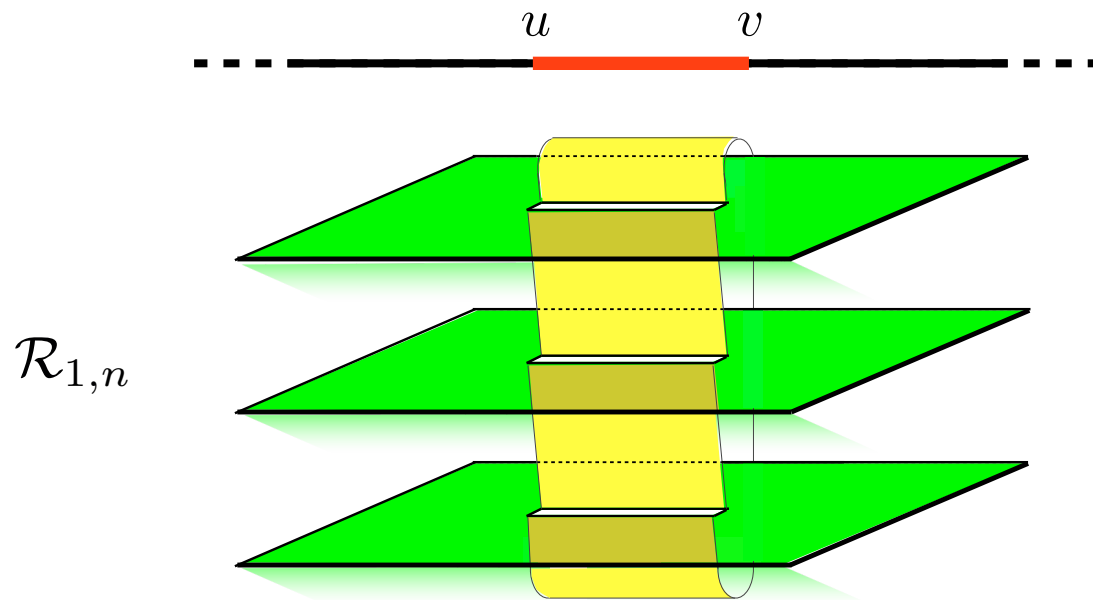
- Global symmetry $i \mapsto i + 1 \bmod n$ from replication

2D CFT: Entanglement entropies as correlation functions

- For one interval ($N = 1$) the Renyi entropies can be written as a two point function of *twist fields* $\mathcal{T}_n, \bar{\mathcal{T}}_n$ on the sphere
 $\mathcal{R}_{1,n}$ can be uniformized into a sphere

[Calabrese, Cardy, JSTAT (2004)]

[Cardy, Castro-Alvaredo, Doyon, J.Stat.Phys. (2008)]



$$\Delta_n = \frac{c}{12} \left(n - \frac{1}{n} \right)$$

$$\text{Tr} \rho_A^n = \frac{\mathcal{Z}_{1,n}}{\mathcal{Z}^n} = \langle \mathcal{T}_n(u) \bar{\mathcal{T}}_n(v) \rangle = \frac{c_n}{|u - v|^{2\Delta_n}}$$

$\Rightarrow$

$$S_A = \frac{c}{3} \log \frac{\ell}{\epsilon} + c'_1$$

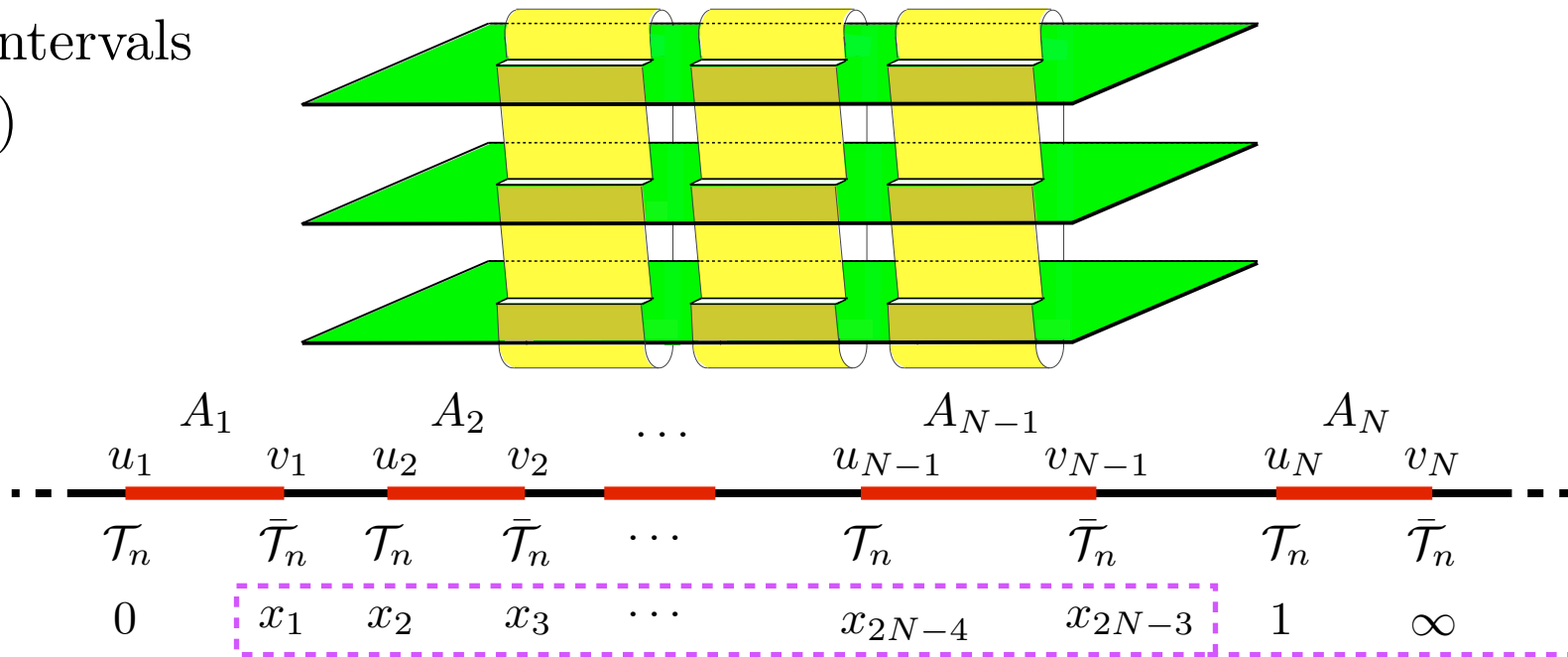
- Twist fields have been studied in string theory

[Zamolodchikov, NPB (1987)] [Dixon, Friedan, Martinec, Shenker, NPB (1987)]

[Knizhnik, CMP (1987)] [Bershadsky, Radul, NPB (1987)]

2D CFT: Entanglement entropies as correlation functions

- Many intervals
($N \geq 2$)



$$\text{Tr} \rho_A^n = \frac{\mathcal{Z}_{N,n}}{\mathcal{Z}^n} = \left\langle \prod_{i=1}^N \mathcal{T}_n(u_i) \bar{\mathcal{T}}_n(v_i) \right\rangle = c_n^N \left| \frac{\prod_{i < j} (u_j - u_i)(v_j - v_i)}{\prod_{i,j} (v_j - u_i)} \right|^{2\Delta_n} \mathcal{F}_{N,n}(\mathbf{x})$$

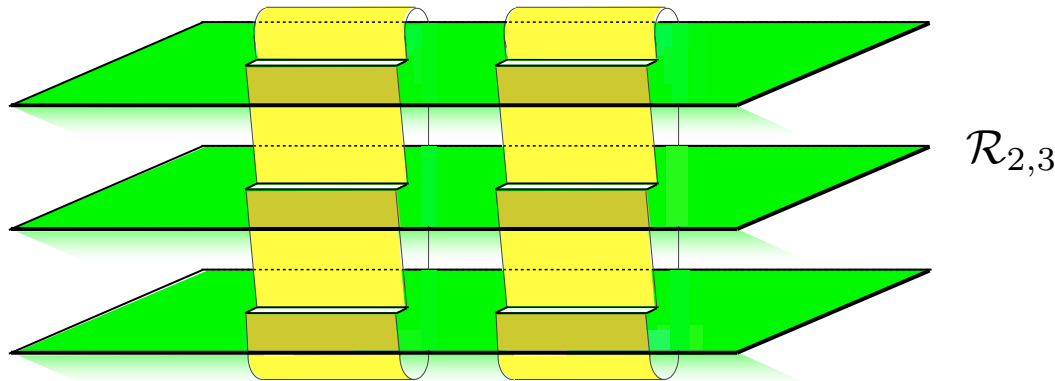
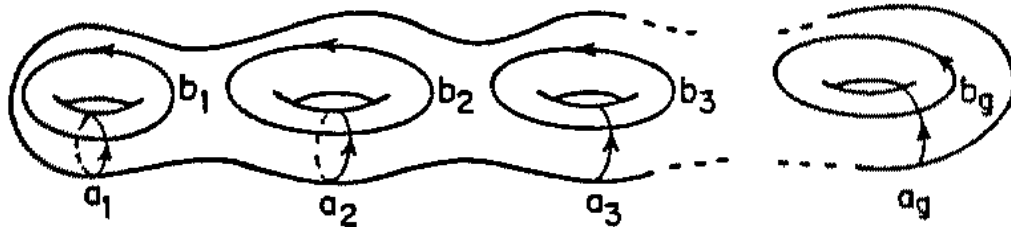
$\mathcal{F}_{N,n}$ can be computed from the partition function on $\mathcal{R}_{N,n}$
genus $g = (N-1)(n-1)$

$$y^n = \prod_{\gamma=1}^N (z - x_{2\gamma-2}) \left[\prod_{\gamma=1}^{N-1} (z - x_{2\gamma-1}) \right]^{n-1}$$

- Two intervals: [Caraglio, Gliozzi, JHEP (2008)] [Furukawa, Pasquier, Shiraishi, PRL (2008)]
[Calabrese, Cardy, E.T., JSTAT (2009), JSTAT (2011)] [Fagotti, Calabrese, JSTAT (2010)]
[Alba, Tagliacozzo, Calabrese, PRB (2009), JSTAT (2011)]

Many intervals: [Coser, Tagliacozzo, E.T., JSTAT (2014)]

Higher genus Riemann surfaces from replication



We are dealing with a subclass of Riemann surfaces of genus $g = n - 1$ obtained from replication

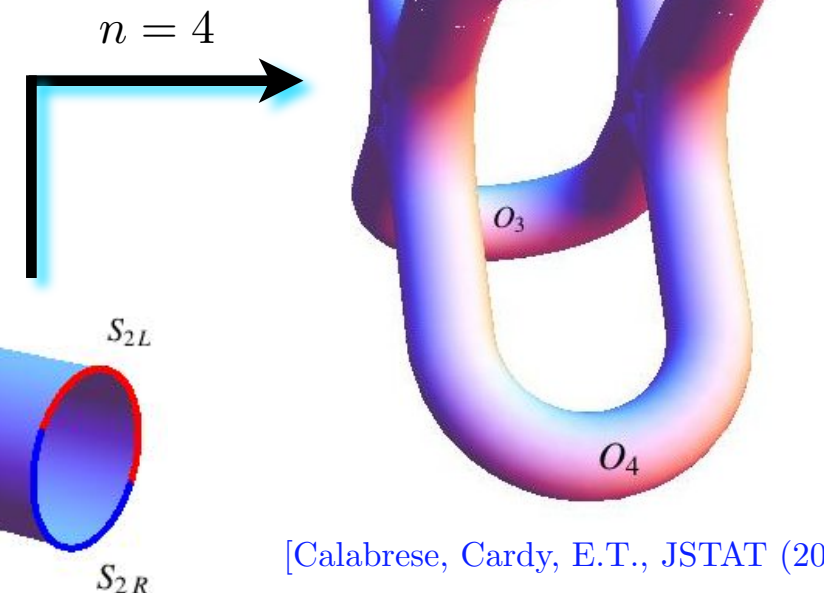
Indeed $\tau = \tau(\mathbf{x})$

A key object is the period matrix τ

$$\oint_{a_i} \omega_j = \delta_{ij} \quad \oint_{b_i} \omega_j = \tau_{ij}$$

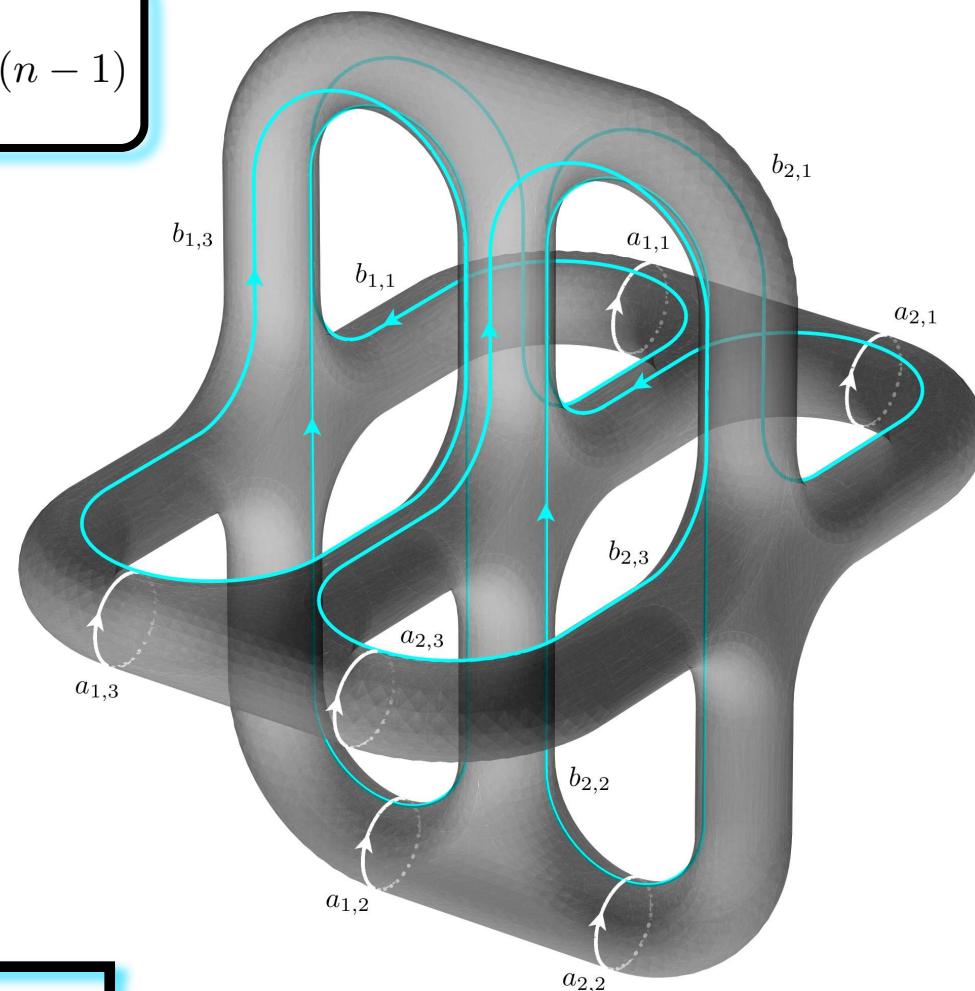
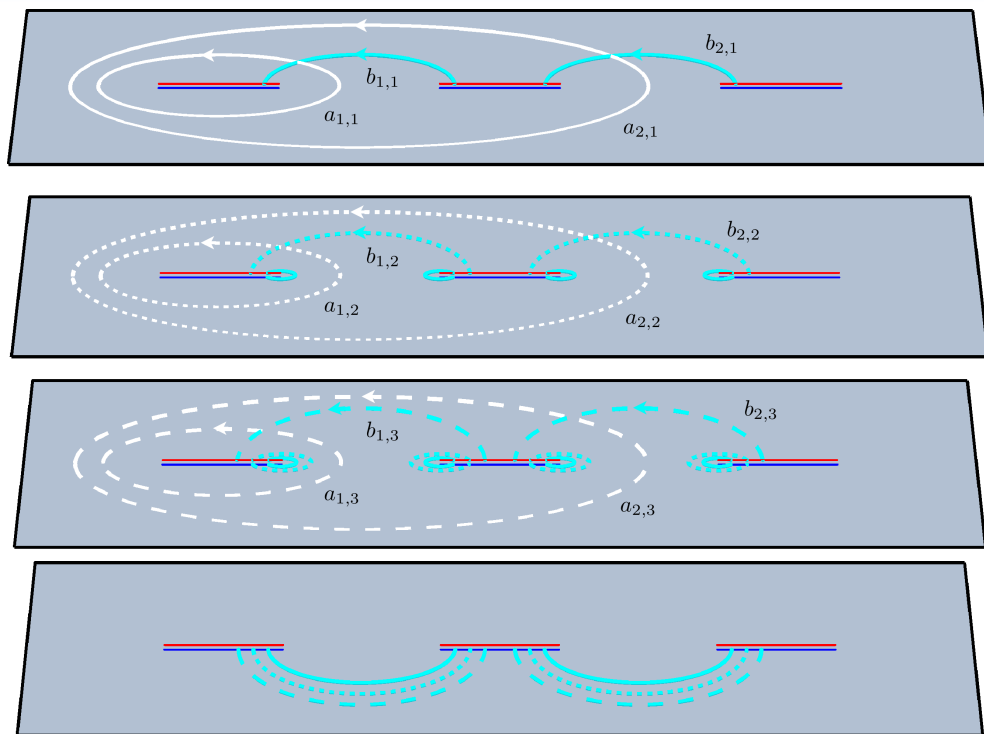
τ is $g \times g$, symmetric and $\text{Im}(\tau) > 0$

$3g - 3$ complex moduli for $g \geq 2$



The period matrix

$$y^n = \prod_{\gamma=1}^N (z - x_{2\gamma-2}) \left[\prod_{\gamma=1}^{N-1} (z - x_{2\gamma-1}) \right]^{n-1} \quad g = (N-1)(n-1)$$



canonical homology basis $\{a_{\alpha,j}, b_{\alpha,j}\}$

$$\omega_{\alpha,j} = \frac{z^{\alpha-1} v(z)^{j-1}}{y^j} dz$$

$$\begin{aligned} \alpha &= 1, \dots, N-1 \\ j &= 1, \dots, n-1 \end{aligned}$$

$$\mathcal{A}_{k,j}^{\beta,\alpha} = \oint_{a_{\alpha,j}} \omega_{\beta,k}$$

$$\mathcal{B}_{k,j}^{\beta,\alpha} = \oint_{b_{\alpha,j}} \omega_{\beta,k}$$

$$\tau = \mathcal{A}^{-1} \cdot \mathcal{B}$$

$$\tau = \tau(\mathbf{x})$$

Free compactified boson

[Coser, Tagliacozzo, E.T., JSTAT (2014)]

■ Compute τ [Enolski, Grava, IMRN (2004)]

■ Riemann theta function ($p \times p$ complex, symmetric matrix Ω with $\text{Im}(\Omega) > 0$)

$$\Theta(\mathbf{0}|\Omega) = \sum_{\mathbf{m} \in \mathbb{Z}^p} \exp(i\pi \mathbf{m}^t \cdot \Omega \cdot \mathbf{m})$$

■ Free compactified boson ($\eta \propto R^2$) $g = (N - 1)(n - 1)$

$$\mathcal{F}_{N,n}(\mathbf{x}) = \frac{\Theta(\mathbf{0}|T_\eta)}{|\Theta(\mathbf{0}|\tau)|^2} \quad T_\eta = \begin{pmatrix} i\eta \mathcal{I} & \mathcal{R} \\ \mathcal{R} & i\mathcal{I}/\eta \end{pmatrix} \quad \tau = \mathcal{R} + i\mathcal{I}$$

○ Invariance $\eta \leftrightarrow 1/\eta$ and $A \leftrightarrow B$

➔ Nasty n dependence

The mutual information can be computed for the uncompactified boson with $N = 2$ [Calabrese, Cardy, E.T., JSTAT (2009)]

■ Analytic checks:

○ Counting of the components of $\mathcal{R}$ based on the dihedral group
[Headrick, Lawrence, Roberts, JSTAT (2012)]

○ The hyperelliptic case ($n = 2$) [Zamolodchikov, NPB (1987)] is recovered

○ The known result for $N = 2$ [Calabrese, Cardy, E.T., JSTAT (2009)] is recovered

The periodic harmonic chain (I)

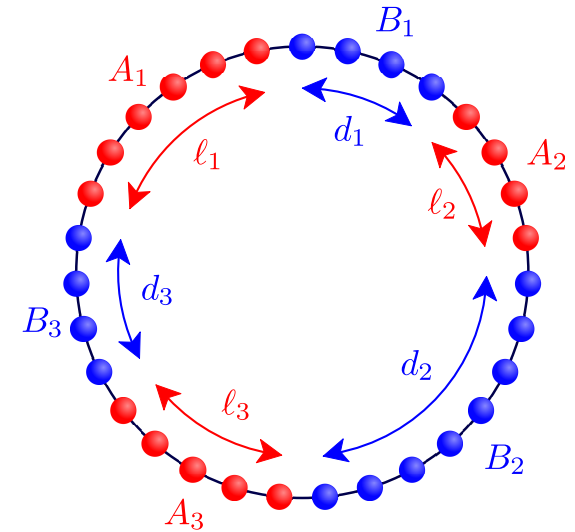
- Periodic chain of harmonic oscillators

$$H = \sum_{n=0}^{L-1} \left(\frac{1}{2M} p_n^2 + \frac{M\omega^2}{2} q_n^2 + \frac{K}{2} (q_{n+1} - q_n)^2 \right)$$

The massless case in the continuum limit is the $c = 1$ free boson on the line

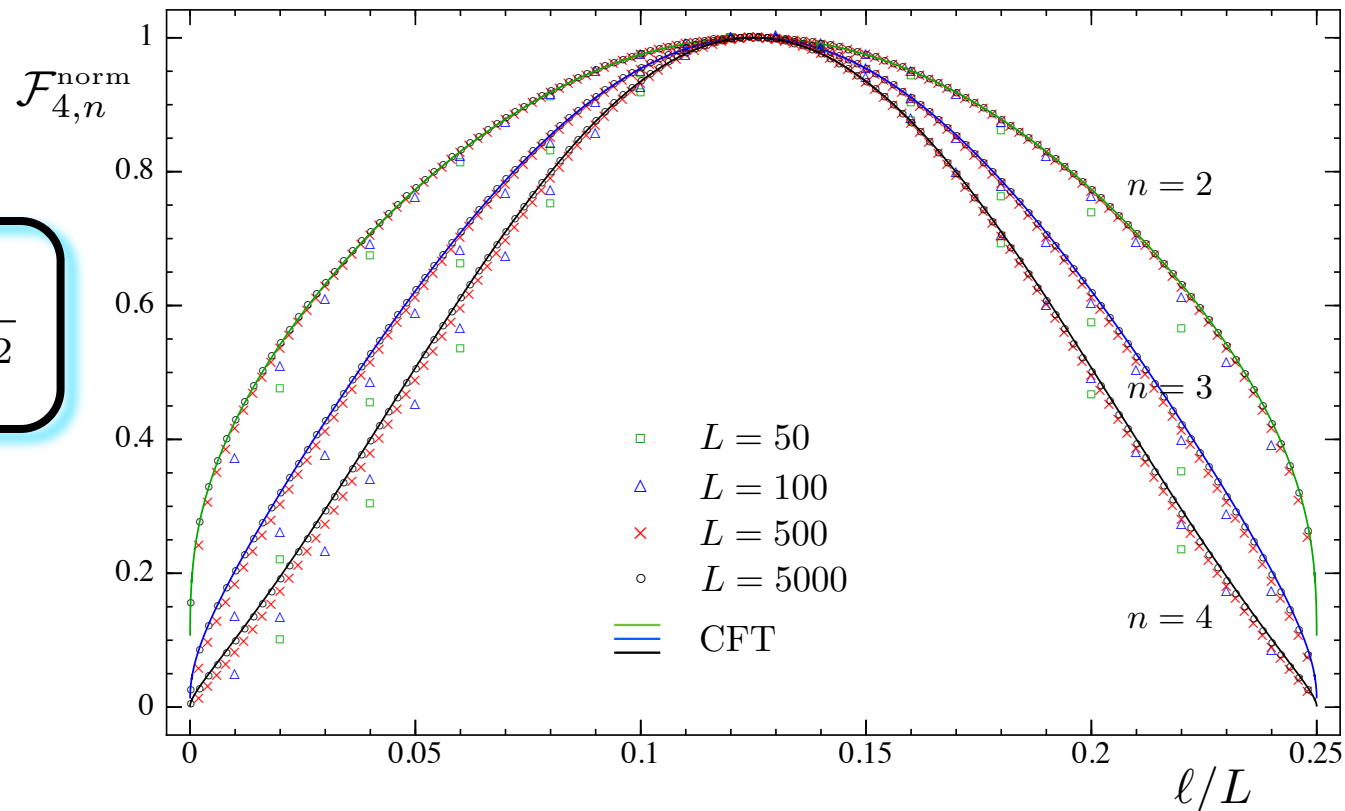
[Peschel, Chung, JPA (1999)] [Botero, Reznik, PRA (2004)]

[Audenaert, Eisert, Plenio, Werner, PRA (2002)]



- Decompactification regime (large η)

$$\mathcal{F}_{N,n}^{\text{dec}}(\mathbf{x}) = \frac{\eta^{g/2}}{\sqrt{\det(\mathcal{I})} |\Theta(\mathbf{0}|\tau)|^2}$$



The periodic harmonic chain (II)

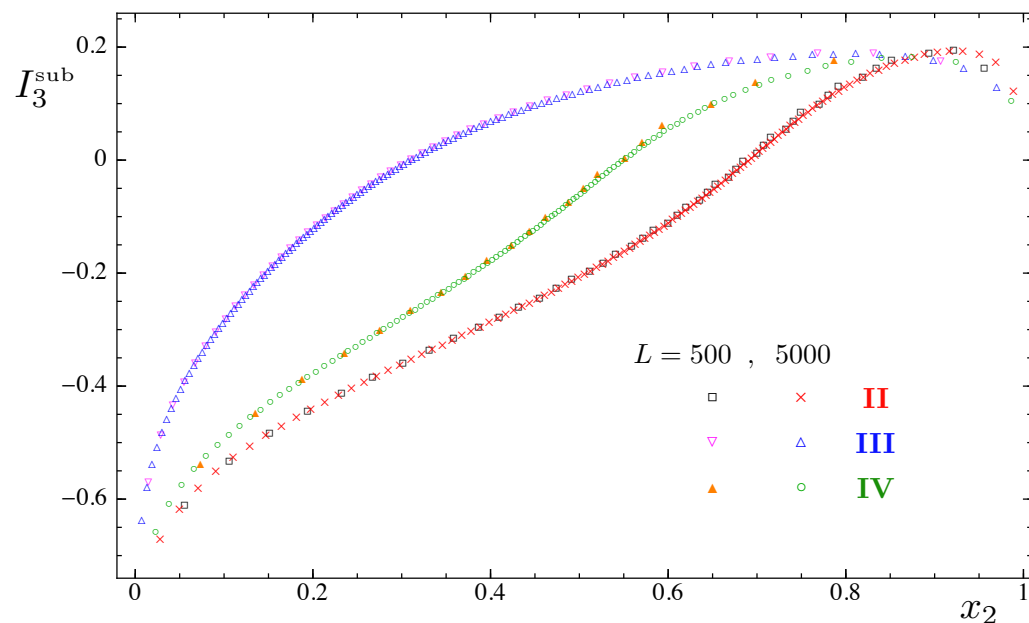
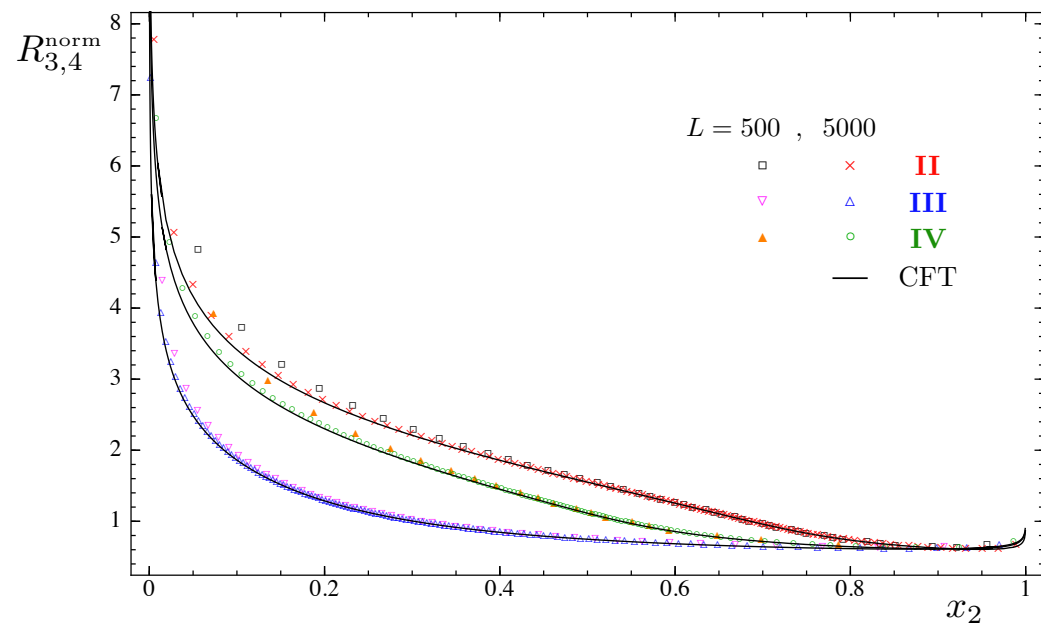
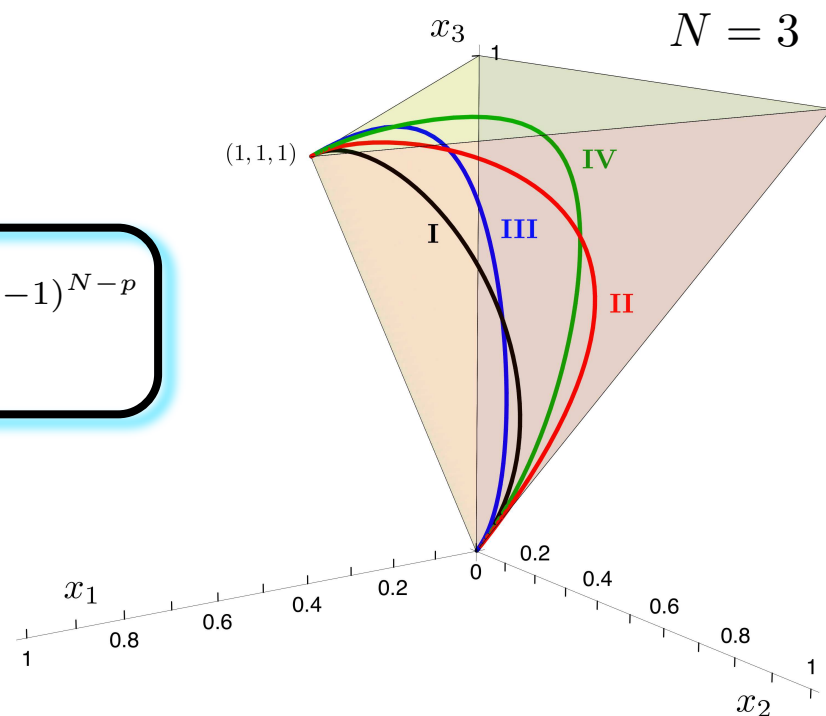
UV finite ratio

$$R_{2,n} = \frac{\text{Tr} \rho_{\{1,2\}}^n}{\text{Tr} \rho_{\{1\}}^n \text{Tr} \rho_{\{2\}}^n} \quad R_{3,n} = \frac{\text{Tr} \rho_{\{1,2,3\}}^n (\text{Tr} \rho_{\{1\}}^n \text{Tr} \rho_{\{2\}}^n \text{Tr} \rho_{\{3\}}^n)}{\text{Tr} \rho_{\{1,2\}}^n \text{Tr} \rho_{\{1,3\}}^n \text{Tr} \rho_{\{2,3\}}^n}$$

$$R_{N,n} \equiv \prod_{p=1}^N \prod_{\sigma_{N,p}} (\text{Tr} \rho_{\sigma_{N,p}}^n)^{(-1)^{N-p}} = \prod_{p=2}^N \prod_{\sigma_{N,p}} [\mathcal{F}_{p,n}(\mathbf{x}^{\sigma_{N,p}})]^{(-1)^{N-p}}$$

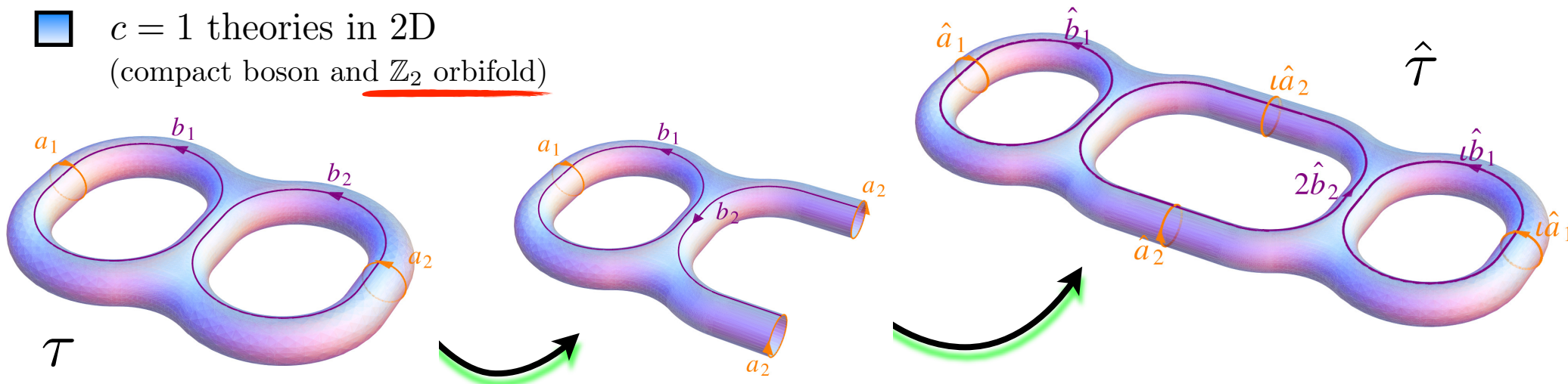
$$I_{A_1, \dots, A_N} \equiv \lim_{n \rightarrow 1} \frac{(-1)^N}{n-1} \log R_{N,n}$$

$I_{A_1, A_2} \equiv S_{A_1} + S_{A_2} - S_{A_1 \cup A_2}$ is the mutual information



The Ising model

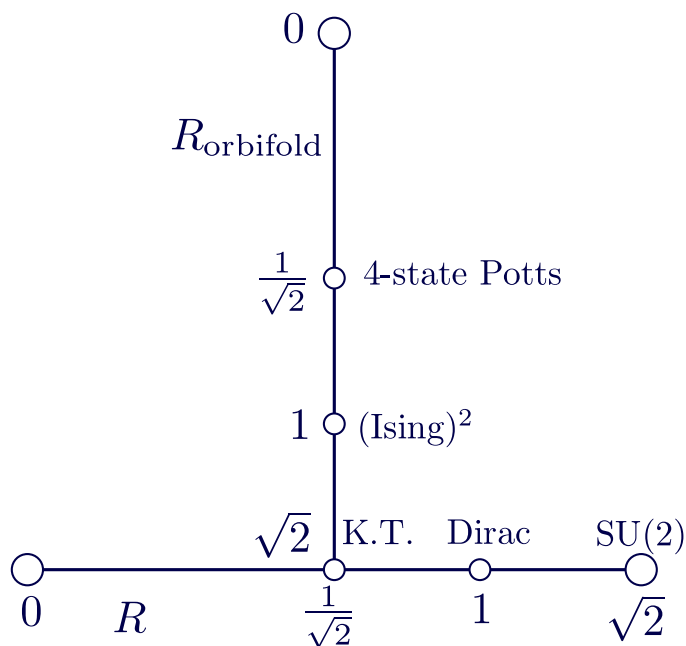
 $c = 1$ theories in 2D
 (compact boson and $\mathbb{Z}_2$ orbifold)



Bosonization on Riemann surfaces

[Alvarez-Gaume, Moore, Vafa; CMP (1986)]

[Dijkgraaf, Verlinde, Verlinde; CMP (1988)]



Ising model

Riemann function with characteristic

$$\Theta[e](\mathbf{0}|\Omega) = \sum_{\mathbf{m} \in \mathbb{Z}^p} \exp [i\pi(\mathbf{m} + \boldsymbol{\varepsilon})^t \cdot \Omega \cdot (\mathbf{m} + \boldsymbol{\varepsilon}) + 2\pi i(\mathbf{m} + \boldsymbol{\varepsilon})^t \cdot \boldsymbol{\delta}]$$

$$\mathcal{F}_{N,n}^{\text{Ising}}(\mathbf{x}) = \frac{\sum_e |\Theta[e](\mathbf{0}|\tau)|}{2^g |\Theta(\mathbf{0}|\tau)|}$$

[Coser, Tagliacozzo, E.T., JSTAT (2014)]



The known result for $N = 2$ is recovered

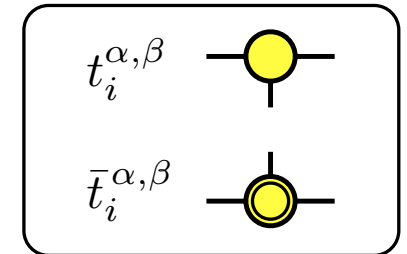
[Calabrese, Cardy, E.T., JSTAT (2011)]

Numerics for the Ising model: Matrix Product States (I)

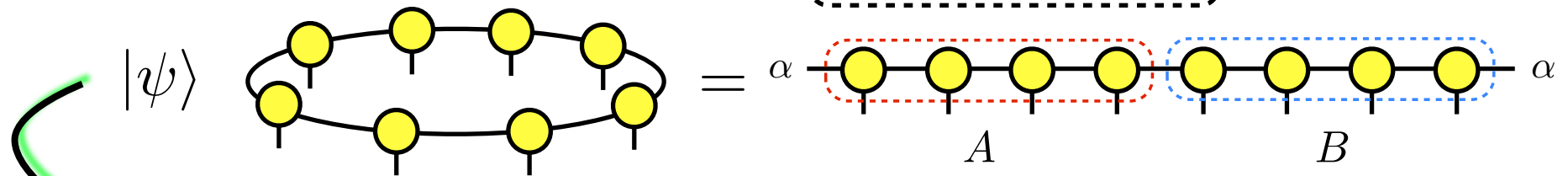
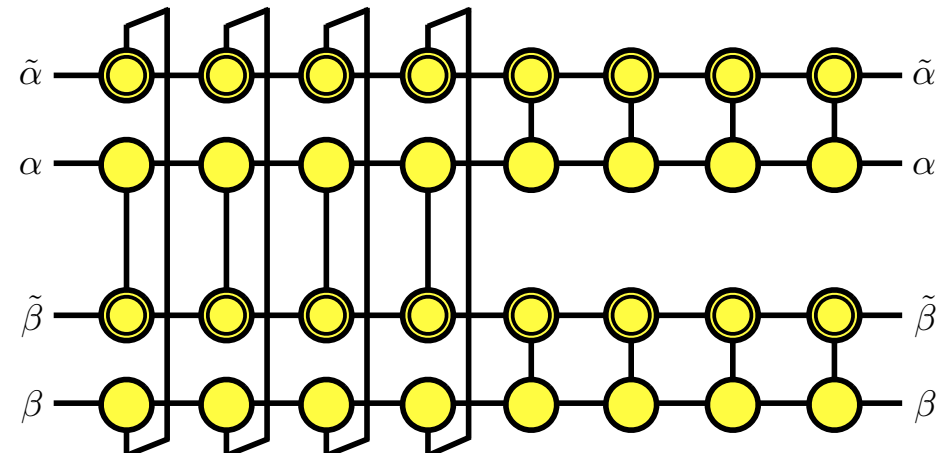
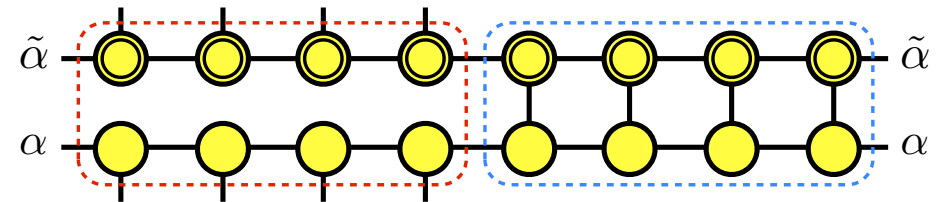
[White, Noack, PRL (1992)] [Ostulid, Rommer, PRL (1995)] [Vidal, PRL (2003)] [Verstraete, Cirac, PRB (2006)]
[Hastings, JSTAT (2007)] ...

Ground state

$$|\Psi\rangle = \sum_{i_1, \dots, i_8=1}^{\delta} \sum_{\alpha_1, \dots, \alpha_8=1}^{\chi} t_{i_1}^{\alpha_1 \alpha_2} t_{i_2}^{\alpha_2 \alpha_3} \dots t_{i_8}^{\alpha_8 \alpha_1} |i_1\rangle |i_2\rangle \dots |i_8\rangle$$



ansatz for $T_{i_1 \dots i_L}$

Reduced density matrix ρ_A 

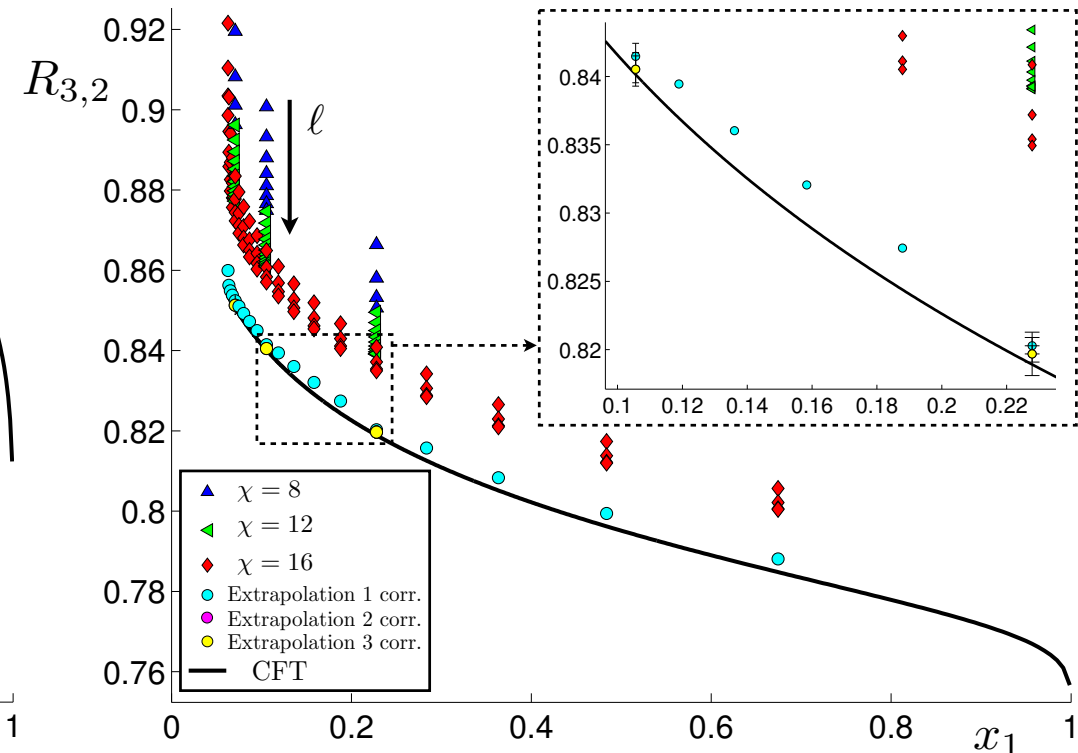
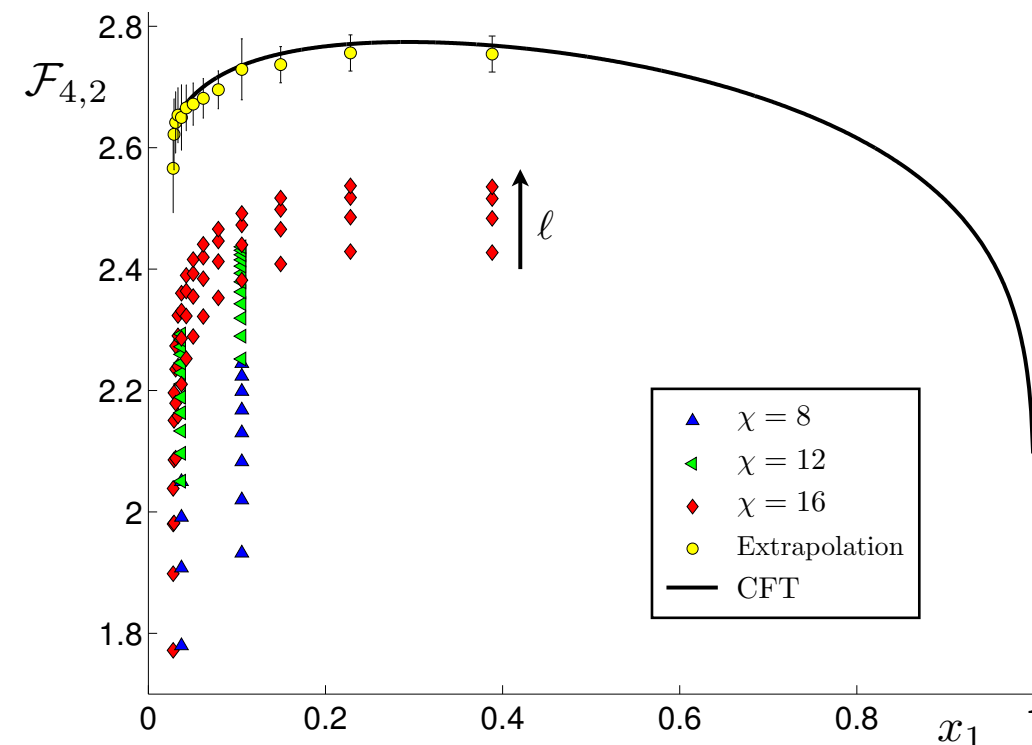
- $t_i^{\alpha\beta}$ are obtained by minimizing $\langle \Psi | H | \Psi \rangle$
- The accuracy of the result depends on the bond dimension χ

Numerics for the Ising model: Matrix Product States (II)

- The $N = 2$ case has been studied numerically through various methods
[Caraglio, Gliozzi, JHEP (2008)] [Furukawa, Pasquier, Shiraishi, PRL (2009)]
[Alba, Tagliacozzo, Calabrese, PRB (2010); JSTAT (2011)]
[Fagotti, Calabrese, JSTAT (2010)]

- For $N > 2$ we considered Ising chain with $30 \leq L \leq 500$.
Variational algorithm of [Pirvu, Verstraete, Vidal, PRB (2011)]
Finite size corrections must be taken into account

[Coser, Tagliacozzo, E.T., JSTAT (2014)]



Short intervals expansion

[Calabrese, Cardy, E.T., JSTAT (2011)]

[Headrick, PRD (2010)]

□ $\text{Tr} \rho_A^n$ when the lengths ℓ_p of the intervals are small w.r.t. to other characteristic lengths of the system

□ $A = [u_1, v_1] \cup [u_2, v_2]$ and r is the distance between the centers

$$\text{Tr} \rho_A^n = c_n^2 (\ell_1 \ell_2)^{-c/6(n-1/n)} \sum_{\{k_j\}} \left(\frac{\ell_1 \ell_2}{n^2 r^2} \right)^{\sum_j (\Delta_j + \bar{\Delta}_j)} \left\langle \prod_{j=1}^n \phi_{k_j} (e^{2\pi i j/n}) \right\rangle_{\mathbf{C}}^2$$

□ $\mathcal{F}_n(x) = 1 + \left(\frac{x}{4n^2} \right)^\alpha s_2(n) + \left(\frac{x}{4n^2} \right)^{2\alpha} s_4(n) + \dots$

$$s_2(n) = \mathcal{N} \frac{n}{2} \sum_{j=1}^{n-1} \frac{1}{[\sin(\pi \frac{j}{n})]^{2\alpha}} \begin{array}{ll} \rightarrow \text{compactified boson} & \alpha = \min[\eta, 1/\eta] \\ \rightarrow \text{Ising} & \alpha = 1/4 \end{array} \quad \begin{array}{ll} \mathcal{N} = 2 \\ \mathcal{N} = 1 \end{array} \quad s'_2(1) = \mathcal{N} \frac{\sqrt{\pi} \Gamma(\alpha + 1)}{4\Gamma(\alpha + \frac{3}{2})}$$

Also $s_4(n)$ is known for the compactified boson and the Ising model

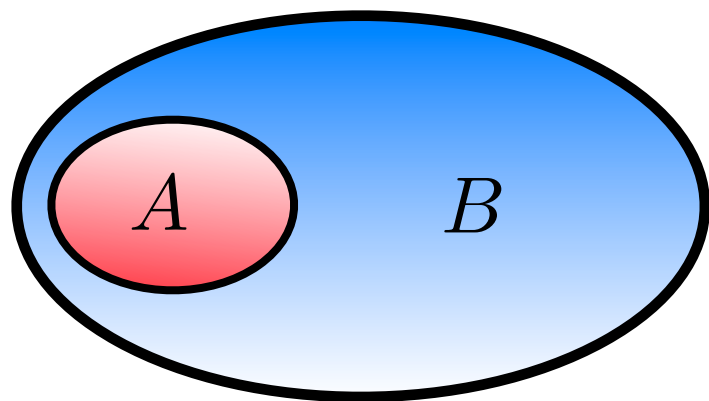
$\text{Tr} \rho_A^n$ for disjoint intervals contains all the data of the CFT (conformal dimensions and OPE coefficients)

The vacuum is not empty

Motivations for Negativity

Quantum system ($\mathcal{H}$) in the ground state $|\Psi\rangle \implies \rho = |\Psi\rangle\langle\Psi|$ ($\text{Tr}\rho^n = 1$)

Bipartite system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$



reduced
density matrix

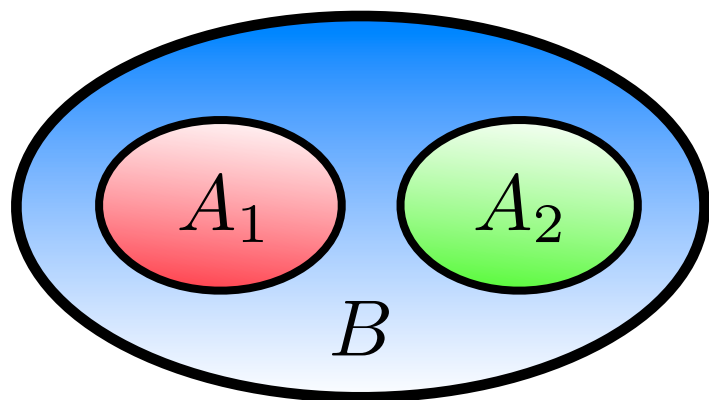
$$\rho_A = \text{Tr}_B \rho$$

Entanglement
entropy

$$S_A = -\text{Tr}_A(\rho_A \log \rho_A)$$

$S_A = S_B$ for pure states

Tripartite system $\mathcal{H} = \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_B \implies \rho_{A_1 \cup A_2}$ is mixed



$S_{A_1 \cup A_2}$: entanglement between $A_1 \cup A_2$ and B

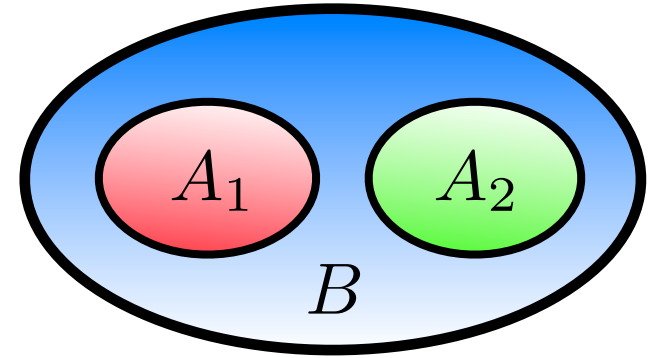
What about the entanglement between A_1 and A_2 ?

A computable measure of the entanglement is the *logarithmic negativity*

Partial transpose & Negativity: definitions

- $\rho = \rho_{A_1 \cup A_2}$ is a mixed state
 $|e_i^{(1)}\rangle$ and $|e_j^{(2)}\rangle$ bases of $\mathcal{H}_{A_1}$ and $\mathcal{H}_{A_2}$

ρ^{T_2} is the partial transpose of ρ



$$\langle e_i^{(1)} e_j^{(2)} | \rho^{T_2} | e_k^{(1)} e_l^{(2)} \rangle = \langle e_i^{(1)} e_l^{(2)} | \rho | e_k^{(1)} e_j^{(2)} \rangle$$

[Peres, PRL (1996)] [Zyczkowski, Horodecki, Sanpera, Lewenstein, PRA (1998)] [Eisert, (2001)]
[Vidal, Werner, PRA (2002)]

- *Trace norm*

$$||\rho^{T_2}|| = \text{Tr}|\rho^{T_2}| = \sum_i |\lambda_i| = 1 - 2 \sum_{\lambda_i < 0} \lambda_i$$

λ_j eigenvalues of ρ^{T_2}
 $\text{Tr} \rho^{T_2} = 1$

Logarithmic negativity

$$\mathcal{E}_{A_2} = \ln ||\rho^{T_2}|| = \ln \text{Tr}|\rho^{T_2}|$$

- Same definition for a bipartite system

$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ in a *generic* state $\rho \longrightarrow \mathcal{E}_A = \mathcal{E}_B$

Replica approach to Negativity

$S_A = - \lim_{n \rightarrow 1} \frac{\partial}{\partial n} \text{Tr} \rho_A^n$

Renyi entropies

$S_n = \frac{\log \text{Tr} \rho_A^n}{1 - n} \xrightarrow{n \rightarrow 1} S_A$

A parity effect for $\text{Tr}(\rho^{T_2})^n$

$\text{Tr}(\rho^{T_2})^{n_e} = \sum_i \lambda_i^{n_e} = \sum_{\lambda_i > 0} |\lambda_i|^{n_e} + \sum_{\lambda_i < 0} |\lambda_i|^{n_e}$

$\text{Tr}(\rho^{T_2})^{n_o} = \sum_i \lambda_i^{n_o} = \sum_{\lambda_i > 0} |\lambda_i|^{n_o} - \sum_{\lambda_i < 0} |\lambda_i|^{n_o}$

Replica approach to negativity $\mathcal{E}_A$

$$\mathcal{E}_A = \log \|\rho^{T_2}\| = \lim_{n_e \rightarrow 1} \log [\text{Tr}(\rho^{T_2})^{n_e}]$$

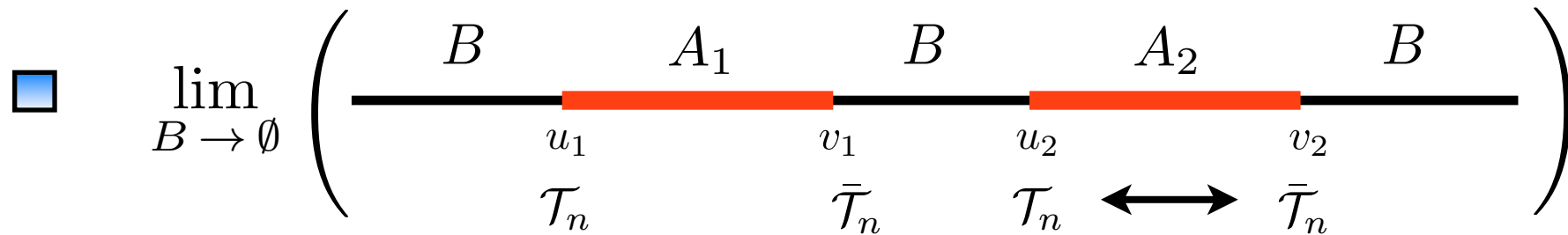
$$\lim_{n_o \rightarrow 1} \text{Tr}(\rho^{T_2})^{n_o} = \text{Tr} \rho^{T_2} = 1$$

Analytic continuation on the even sequence $\text{Tr}(\rho^{T_2})^{n_e}$ (make 1 an even number)

Partial Transposition for bipartite systems: pure states (I)

[Calabrese, Cardy and E.T.; PRL (2012), JSTAT (2013)]

$$\mathcal{H} = \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$$



Partial Transposition = exchange two twist fields

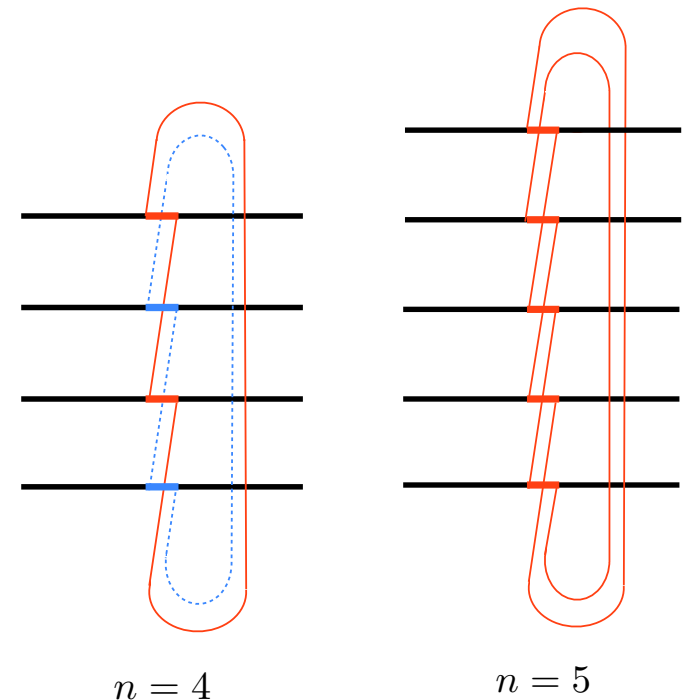
$$\text{Tr}(\rho_A^{T_2})^n = \langle \mathcal{T}_n^2(u_2) \bar{\mathcal{T}}_n^2(v_2) \rangle$$

□ $\mathcal{T}_n^2$ connects the j -th sheet with the $(j+2)$ -th one

□ Even $n = n_e \implies$ decoupling

$$\text{Tr}(\rho_A^{T_2})^{n_e} = (\langle \mathcal{T}_{n_e/2}(u_2) \bar{\mathcal{T}}_{n_e/2}(v_2) \rangle)^2 = \left(\text{Tr} \rho_{A_2}^{n_e/2} \right)^2$$

$$\text{Tr}(\rho_A^{T_2})^{n_o} = \langle \mathcal{T}_{n_o}(u_2) \bar{\mathcal{T}}_{n_o}(v_2) \rangle = \text{Tr} \rho_{A_2}^{n_o}$$



Partial Transpose: pure states (II)

□ $\rho = |\Psi\rangle\langle\Psi| \quad \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$
 $\Rightarrow$ Schmidt decomposition $|\Psi\rangle = \sum_k c_k |\Psi_k\rangle_1 |\Psi_k\rangle_2 \quad c_k \geq 0 \quad \sum_k c_k^2 = 1$

$$\text{Tr}(\rho^{T_2})^n = \begin{cases} \text{Tr} \rho_2^n & n = n_o \quad \text{odd} \\ (\text{Tr} \rho_2^{n/2})^2 & n = n_e \quad \text{even} \end{cases}$$

□ Two dimensional CFTs

$$\Delta_{\mathcal{T}_{n_e}^2} = \frac{c}{6} \left(\frac{n_e}{2} - \frac{2}{n_e} \right)$$

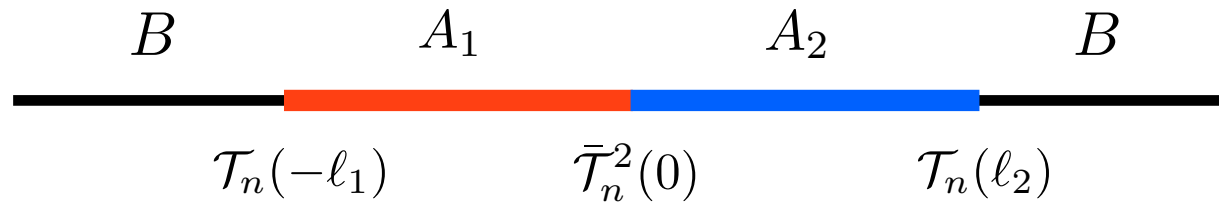
$$\Delta_{\mathcal{T}_{n_o}^2} = \frac{c}{12} \left(n_o - \frac{1}{n_o} \right) = \Delta_{\mathcal{T}_{n_o}}$$

↓

$$\|\rho_A^{T_2}\| = \lim_{n_e \rightarrow 1} \text{Tr}(\rho_A^{T_2})^{n_e} \propto \ell^{c/2} \longrightarrow \mathcal{E} = \frac{c}{2} \ln \ell + \text{const}$$

□ For $n_e \rightarrow 1$ we find $\mathcal{E} = 2 \log \text{Tr} \rho_2^{1/2}$ (Renyi entropy 1/2)

Partial Transpose in 2D CFT: two adjacent intervals



Three point function

$$\text{Tr}(\rho_A^{T_2})^n = \langle \mathcal{T}_n(-\ell_1) \bar{\mathcal{T}}_n^2(0) \mathcal{T}_n(\ell_2) \rangle$$

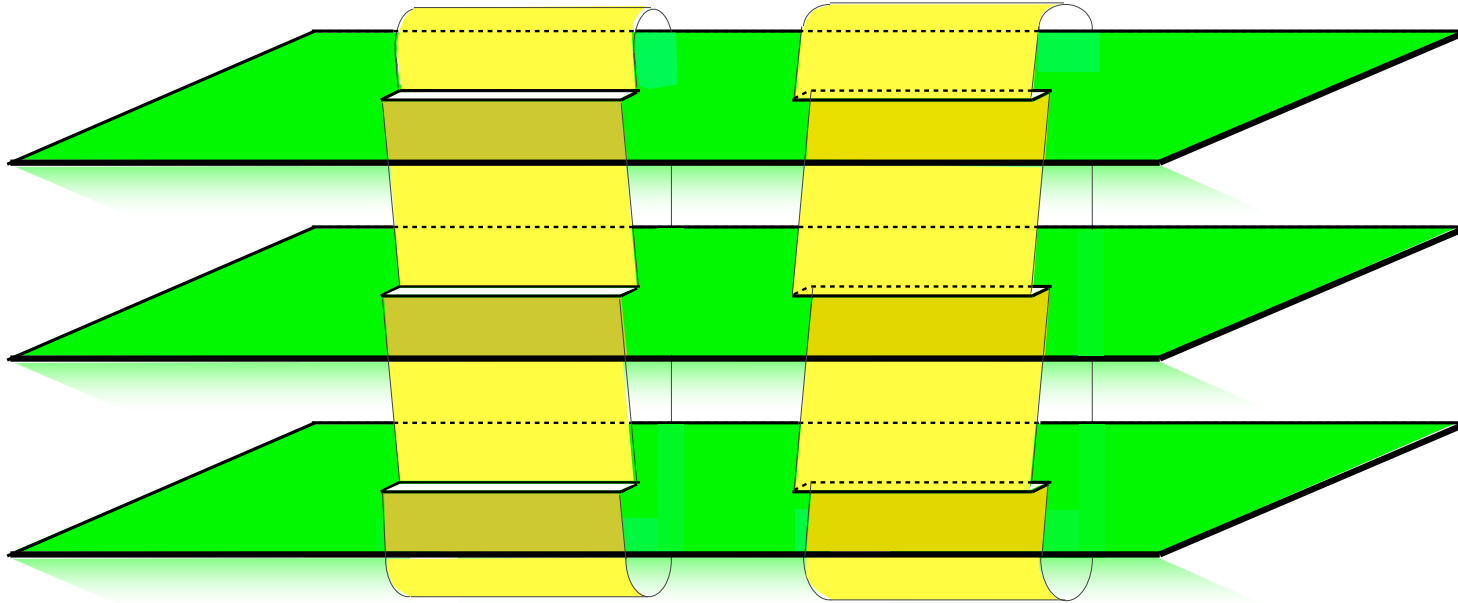
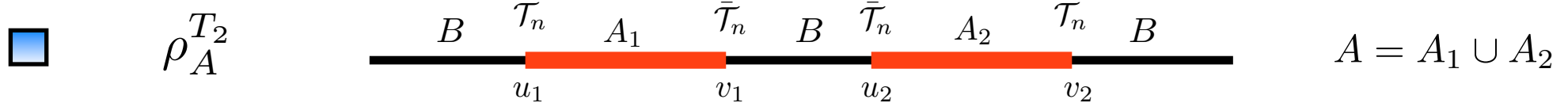
$$\text{Tr}(\rho_A^{T_2})^{n_e} \propto (\ell_1 \ell_2)^{-\frac{c}{6}(\frac{n_e}{2} - \frac{2}{n_e})} (\ell_1 + \ell_2)^{-\frac{c}{6}(\frac{n_e}{2} + \frac{1}{n_e})}$$

$$\text{Tr}(\rho_A^{T_2})^{n_o} \propto (\ell_1 \ell_2 (\ell_1 + \ell_2))^{-\frac{c}{12}(n_o - \frac{1}{n_o})}$$

From the *analytic continuation* $n_e \rightarrow 1$

$$\|\rho_A^{T_2}\| \propto \left(\frac{\ell_1 \ell_2}{\ell_1 + \ell_2} \right)^{c/4} \longrightarrow \mathcal{E} = \frac{c}{4} \ln \left(\frac{\ell_1 \ell_2}{\ell_1 + \ell_2} \right) + \text{const}$$

Partial Transpose in 2D CFT: two disjoint intervals




 Four point function of twist fields in the complex plane (free boson, uncompactified)

$$\frac{\text{Tr}(\rho_A^{T_2})^n}{\text{Tr} \rho_A^n} = \left[\frac{(1-y)^{\frac{2}{3}(n-\frac{1}{n})} \prod_{k=1}^{n-1} F_{\frac{k}{n}}(y) F_{\frac{k}{n}}(1-y)}{\prod_{k=1}^{n-1} \text{Re}\left(F_{\frac{k}{n}}\left(\frac{y}{y-1}\right) \bar{F}_{\frac{k}{n}}\left(\frac{1}{1-y}\right)\right)} \right]^{\frac{1}{2}}$$

$F_\beta(y) \equiv {}_2F_1(\beta, 1-\beta; 1; y)$
 $0 < y < 1$

Free compactified boson: traces of the partial transpose

□ We need to extend $\mathcal{F}_{N=2,n}$ to $x \in \mathbb{C}$

$$F_y(x) \equiv {}_2F_1(y, 1-y; 1; x)$$

$$\mathcal{F}_n(x, \bar{x}) = \frac{\Theta(0|T)}{\prod_{k=1}^{n-1} |F_{k/n}(x)|}$$

$$\Theta(0|\Gamma) \equiv \sum_{m \in \mathbf{Z}^G} \exp [i\pi m^t \cdot \Gamma \cdot m]$$

$$T = \begin{pmatrix} i\eta\mathcal{I} & \mathcal{R} \\ \mathcal{R} & i\mathcal{I}/\eta \end{pmatrix}$$

where the *period matrix* is

$$\tau_{rs} \equiv \mathcal{R}_{rs} + i\mathcal{I}_{rs} = \frac{2}{n} \sum_{k=1}^{n-1} \sin(\pi k/n) \left[i \frac{F_{k/n}(1-x)}{F_{k/n}(x)} \right] \cos(2\pi k/n(r-s))$$

$$r, s = 1, \dots, n-1$$

 nasty n dependence

➔ $\mathcal{F}_n(x, \bar{x})$ is invariant under $\eta \leftrightarrow 1/\eta$ and under $x \leftrightarrow 1-x$

➔ $\mathcal{F}_n(x, \bar{x})$ gives the previous result for $x \in (0, 1)$

Renyi entropies vs traces of the Partial Transpose

$$\text{Tr } \rho_A^n$$



$$\text{Tr}(\rho_A^{T_2})^n$$



Harmonic chains (periodic b.c.) : two adjacent intervals

Harmonic chain on a circle

$$H = \frac{1}{2} \sum_{j=1}^L [p_j^2 + \omega^2 q_j^2 + (q_{j+1} - q_j)^2]$$

Critical regime $\omega = 0$

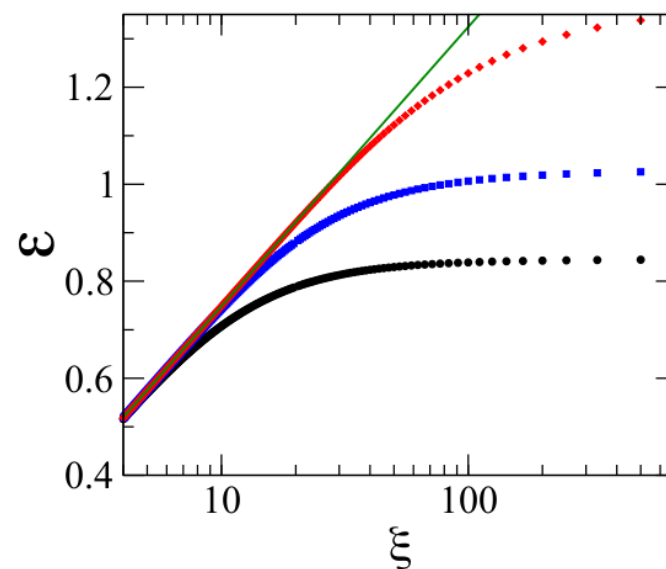
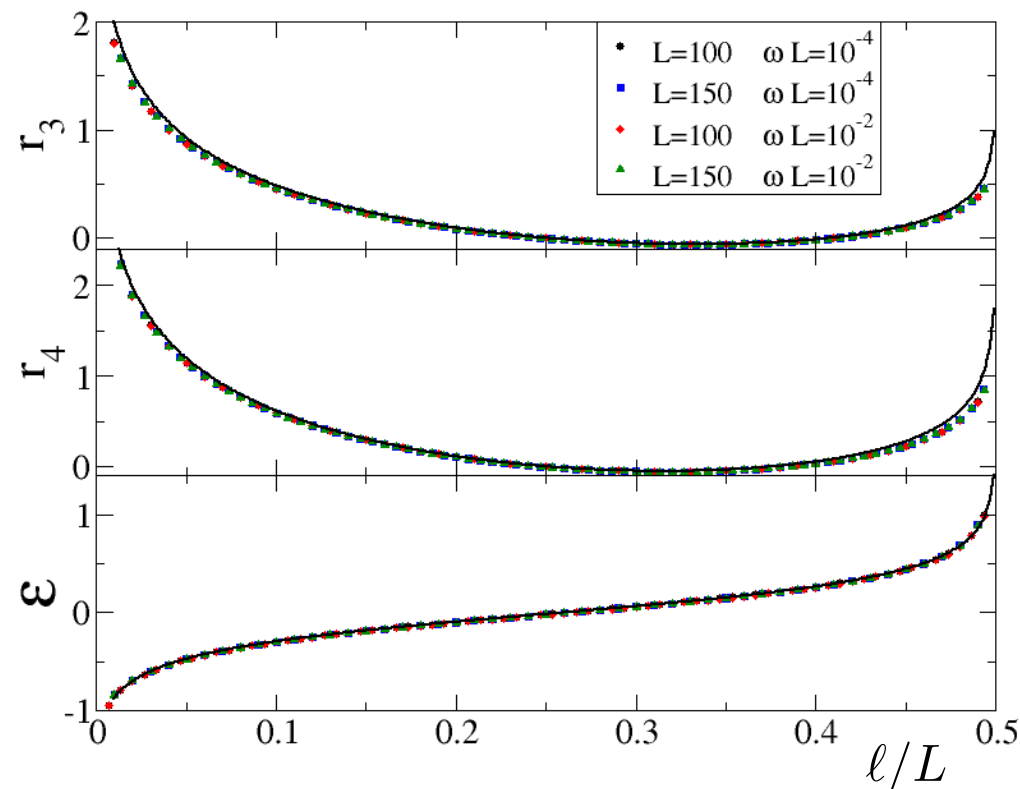
Continuum limit $\longrightarrow$ $c = 1$ free boson

Two adjacent intervals

Finite system: $\ell \longrightarrow \frac{L}{\pi} \sin \frac{\pi \ell}{L}$

$$r_n = \ln \frac{\text{Tr}(\rho_A^{T_{A_2=\ell}})^n}{\text{Tr}(\rho_A^{T_{A_2=L/4}})^n}$$

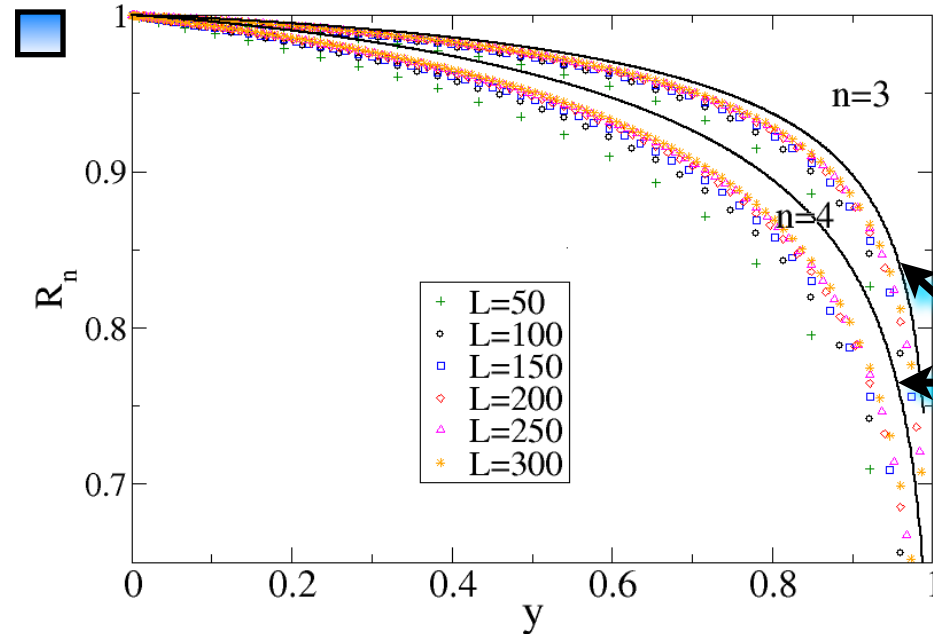
Open chains with Dirichlet b.c. and the massive case $\longrightarrow$ $\mathcal{E} = \frac{c}{4} \log \xi + \dots$
have been studied



Harmonic chains (periodic b.c.) : two disjoint intervals

Previous numerical results for $\mathcal{E}$:
Ising (DMRG) and harmonic chains

[Wichterich, Molina-Vilaplana, Bose, PRB (2009)]
[Marcovitch, Retzker, Plenio, Reznik, PRA (2009)]



[Calabrese, Cardy, E.T.; PRL (2012), JSTAT (2013)]

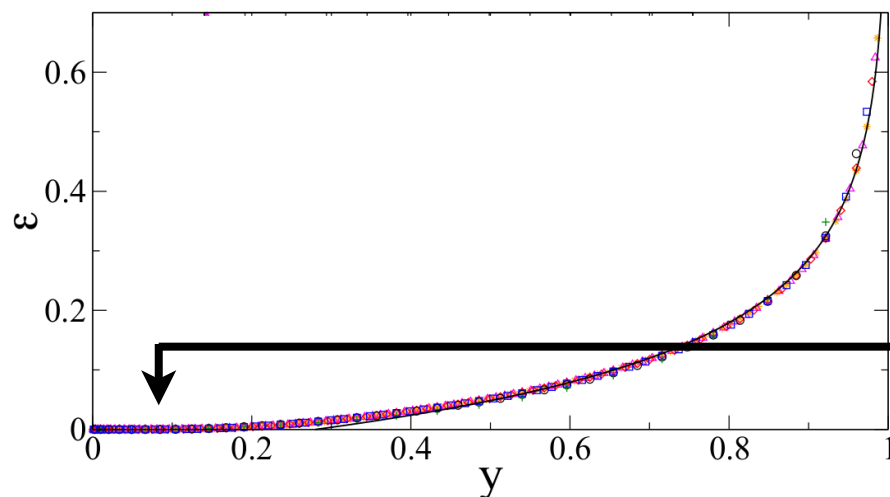
Two disjoint intervals

$$R_n = \frac{\text{Tr}(\rho_A^{T_2})^n}{\text{Tr} \rho_A^n}$$

CFT curves

Analytic continuation for $y \sim 1$

$$\mathcal{E} = -\frac{1}{4} \log(1 - y) + \log K(y) + \text{cnst}$$



Analytic continuation $n_e \rightarrow 1$
for $0 < y < 1$ is missing

It goes to zero faster than any power

Ising model: traces of the partial transpose (CFT & TTN)

Two disjoint intervals

Monte-Carlo analysis

[Alba, JSTAT (2013)]

Tree Tensor Network

[Calabrese, Tagliacozzo and E.T., JSTAT (2013)]

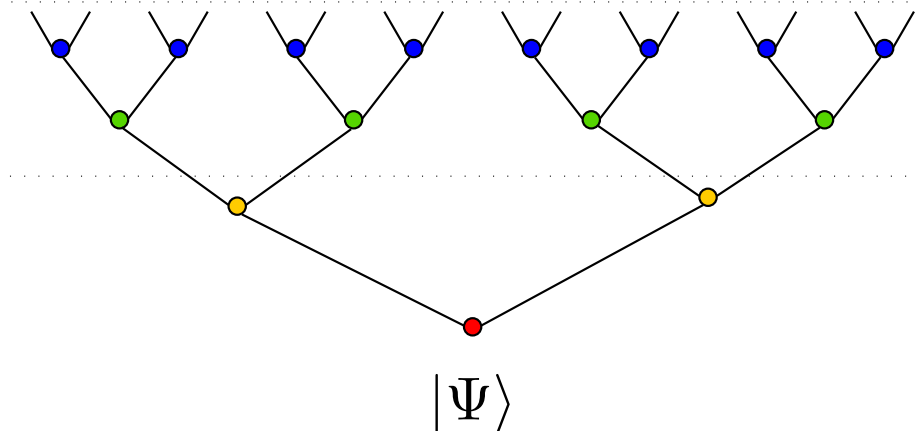
CFT

$$\mathcal{G}_n(y) = (1-y)^{(n-1/n)/6} \frac{\sum_{\mathbf{e}} |\Theta[\mathbf{e}](0|\tau(\frac{y}{y-1}))|}{2^{n-1} \prod_{k=1}^{n-1} |F_{k/n}(\frac{y}{y-1})|^{1/2}}$$

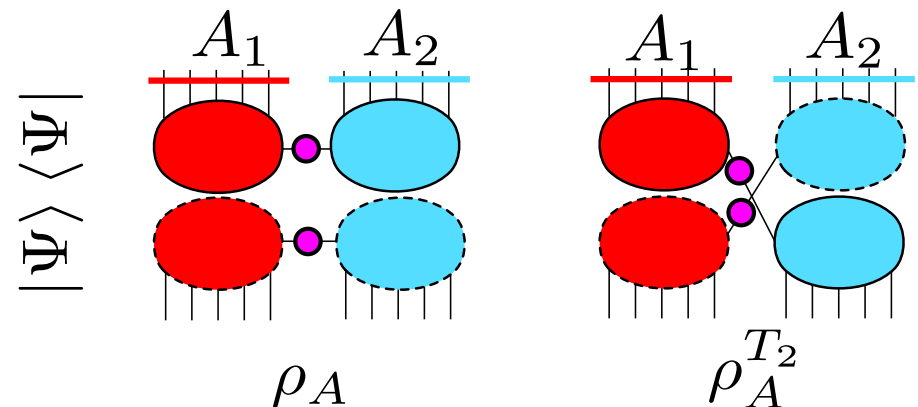
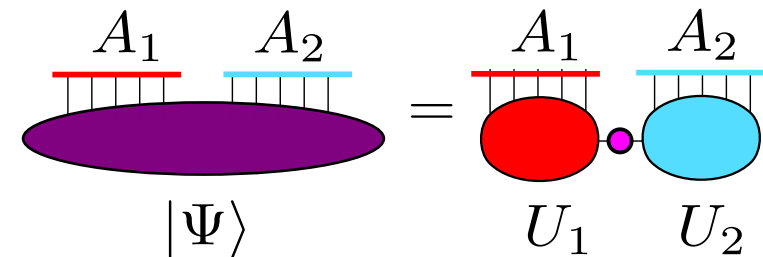
$$0 < y < 1$$

$|\Psi\rangle = \sum_{i_1=1}^d \sum_{i_2=1}^d \cdots \sum_{i_L=1}^d T_{i_1 i_2 \dots i_L} |i_1\rangle |i_2\rangle \cdots |i_L\rangle$

TTN The state is encoded in the network made by smaller tensors

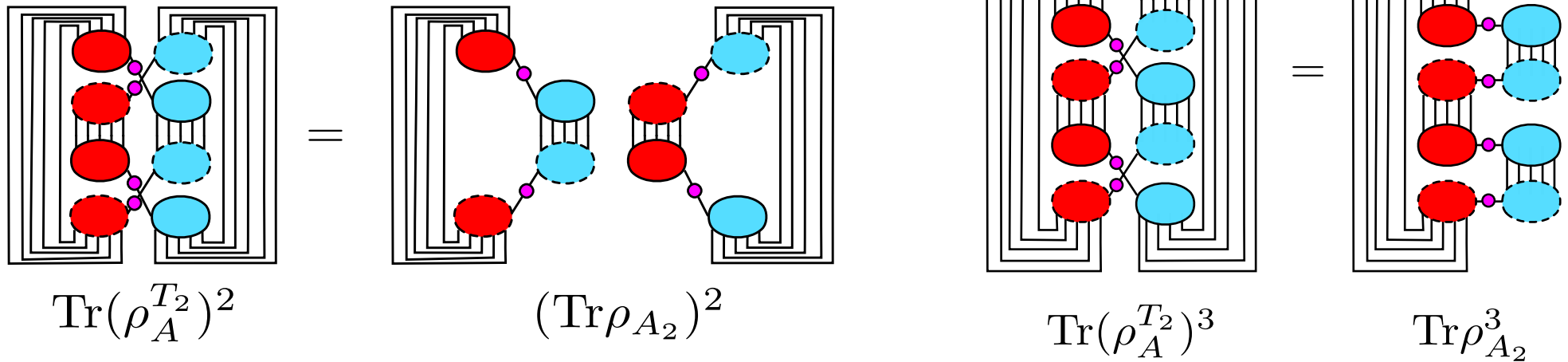


Schmidt decomposition

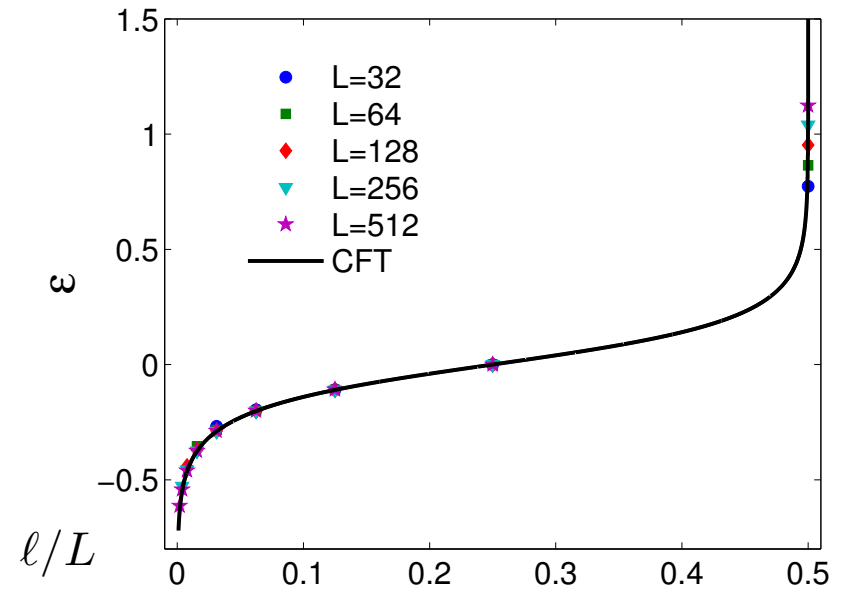
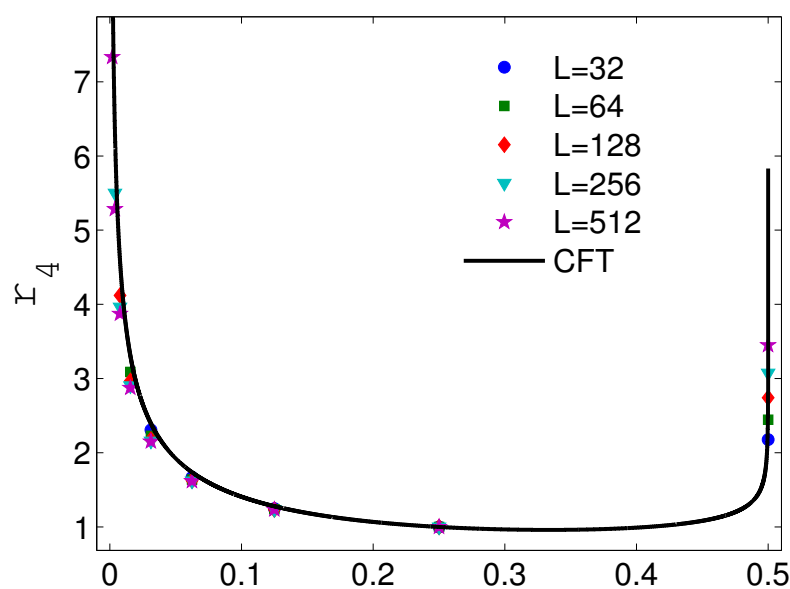


Ising model: partial transpose with Tree Tensor Networks

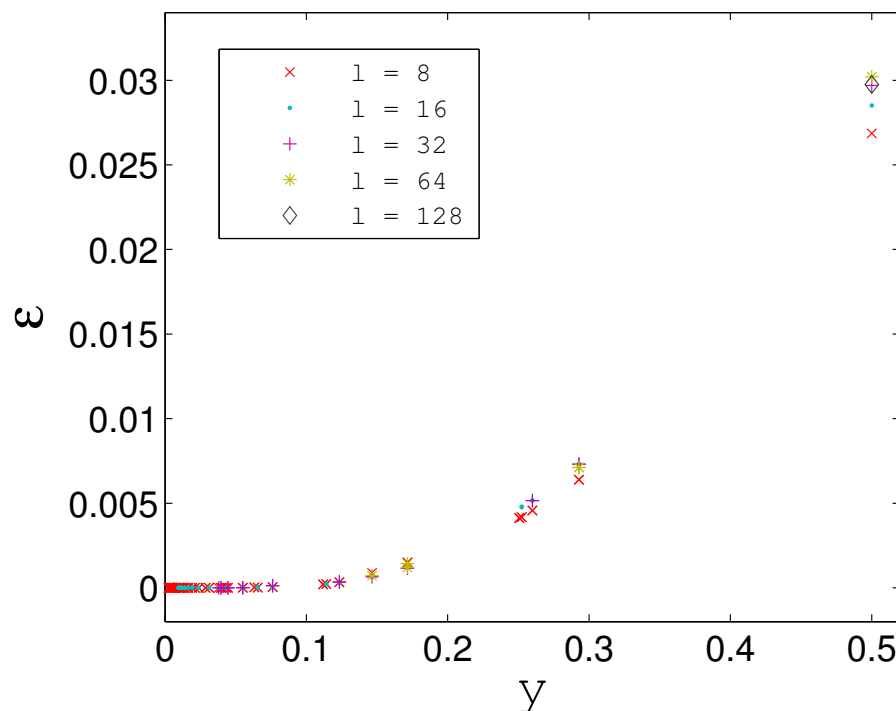
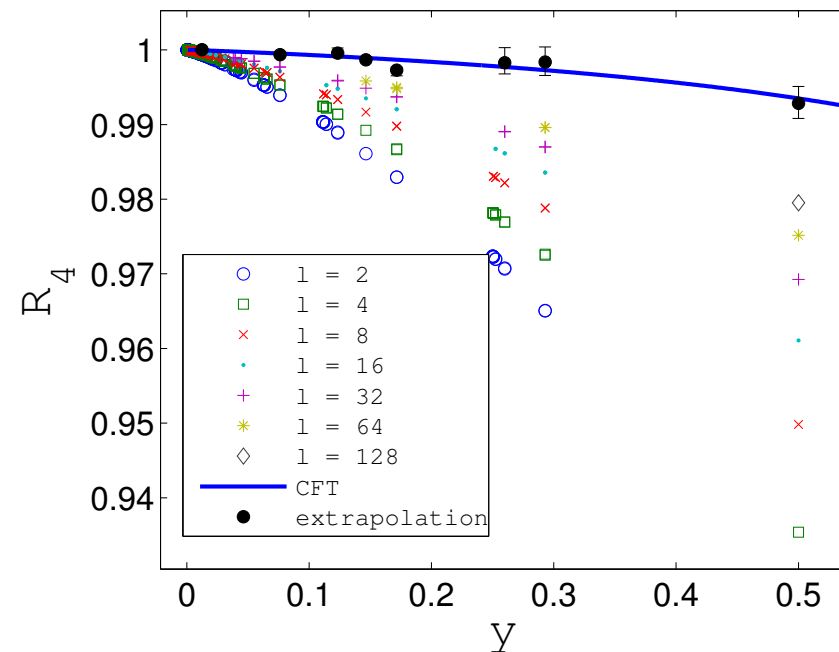
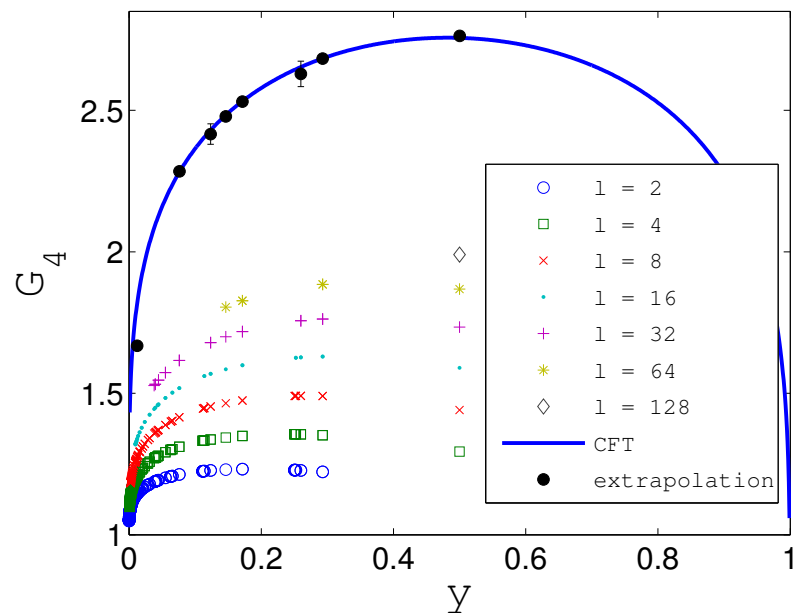
■ Pure states: relation with Renyi entropies in the TTN language



■ Numerical checks:
Two adjacent intervals



Partial transpose with TTN: two disjoint intervals



Logarithmic negativity:
CFT formula still unknown

Holographic entanglement entropy

AdS_{d+2}/CFT_{d+1} correspondence

■ Prescription: in regularized AdS_{d+2}

○ Constant time slice

○ Consider the surfaces γ_A s.t. $\partial\gamma_A = \partial A$ which are homologous to A

(it exists a bulk region $\mathcal{R}$ s.t. $\partial\mathcal{R} = A \cup \gamma_A$)

[Headrick, Takayanagi, PRD (2007)] [Fursaev, JHEP (2006)]

[Azeyanagi, Nishioka, Takayanagi, PRD (2008)]

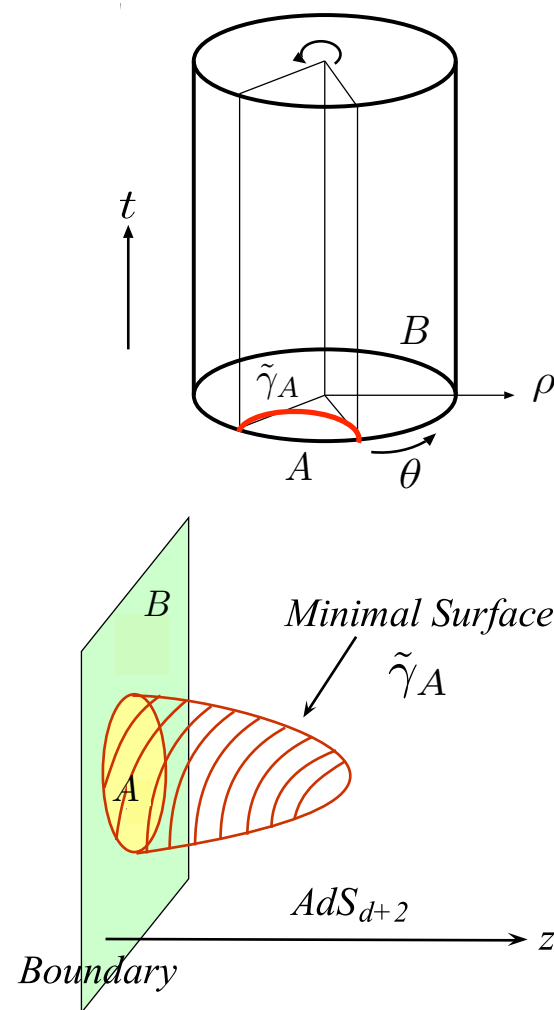
○ Find the surface $\tilde{\gamma}_A$ with minimal area

$$S_A = \frac{\text{Area}(\tilde{\gamma}_A)}{4G_N^{(d+2)}}$$

■ $d = 1$ formula $S_A = \frac{c}{3} \log \frac{\ell}{a} + c_0$ and the area law $S_A \propto \frac{\text{Area}(\partial A)}{a^{d-1}}$

■ Strong subadditivity is satisfied [Headrick, Takayanagi, PRD (2007)]

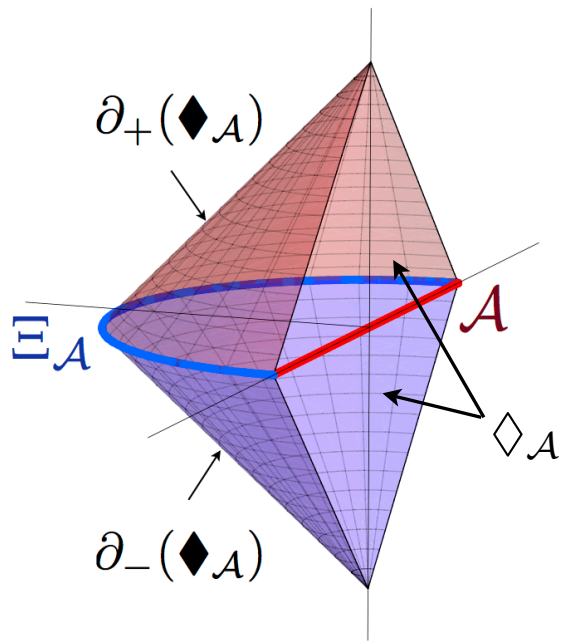
[Ryu, Takayanagi, PRL, JHEP (2006)]



Causal holographic information

[Hubeny, Takayanagi, Rangamani, JHEP (2007)]

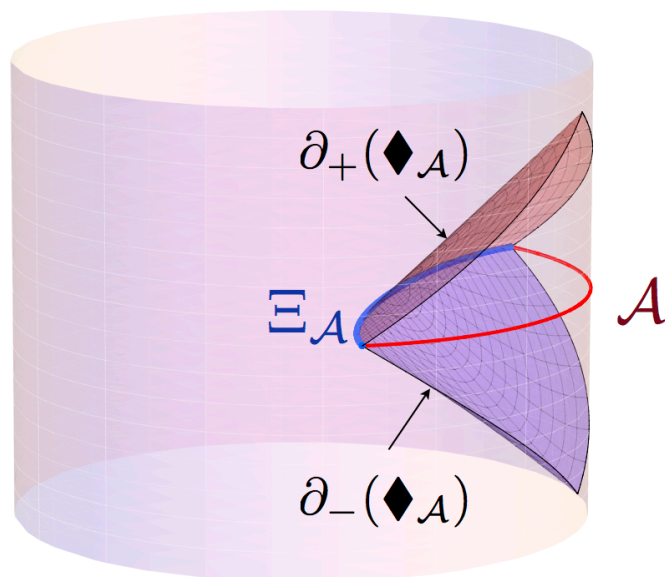
[Hubeny, Rangamani, JHEP (2012)]



- Does holography suggest new quantities to consider?
- Natural construction starting from the *domain of dependence* of A ($\diamond_A$) *causal wedge* $\diamond_A$ associated to $\mathcal{A}$

$$\partial \diamond_A = \partial_+(\diamond_A) \cup \partial_-(\diamond_A)$$

$$\Xi_A = \partial_+(\diamond_A) \cap \partial_-(\diamond_A)$$



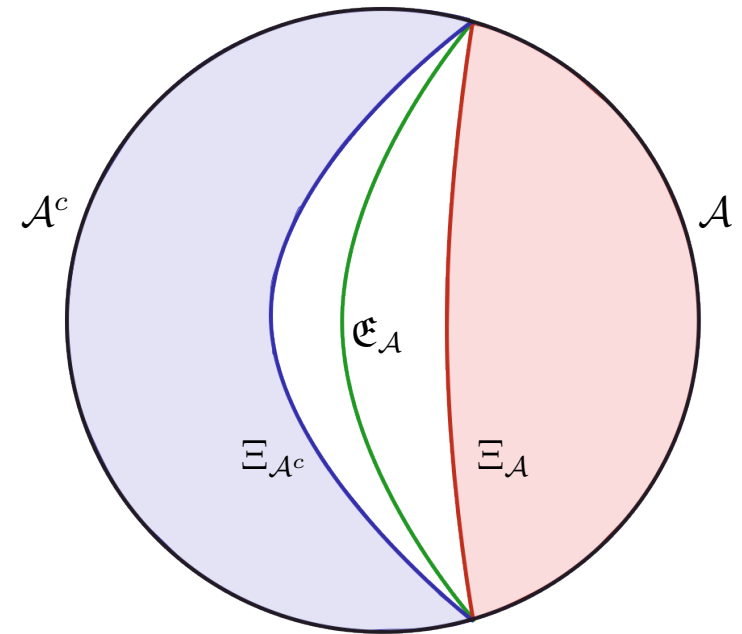
$$\chi_A = \frac{\text{Area}(\Xi_A)}{4G_N^{(d+2)}}$$

Causal holographic information

[Hubeny and Rangamani, JHEP (2012)]

Properties of χ_A

- $\chi_A = S_A$ for AdS_3 , BTZ and rotating BTZ black hole
- $\chi_A = S_A$ for spherical A in the boundary CFT_{d+1}
- for the complementar region: $\chi_{A^c} \neq \chi_A$
- χ_A does not satisfy strong subadditivity

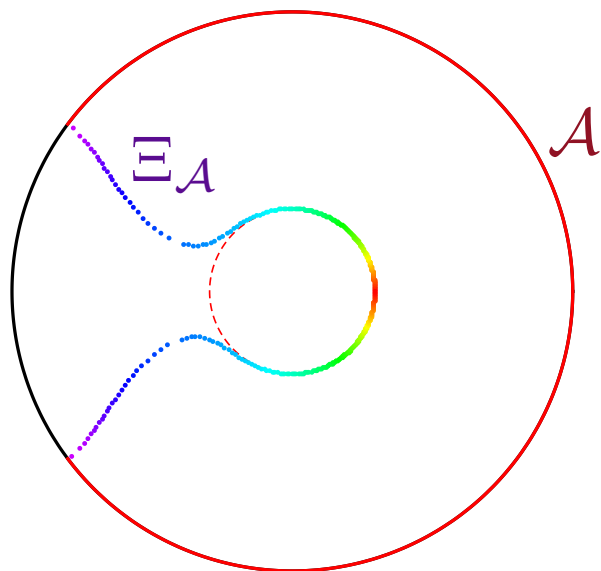


CHI: Schwarzschild-AdS black hole

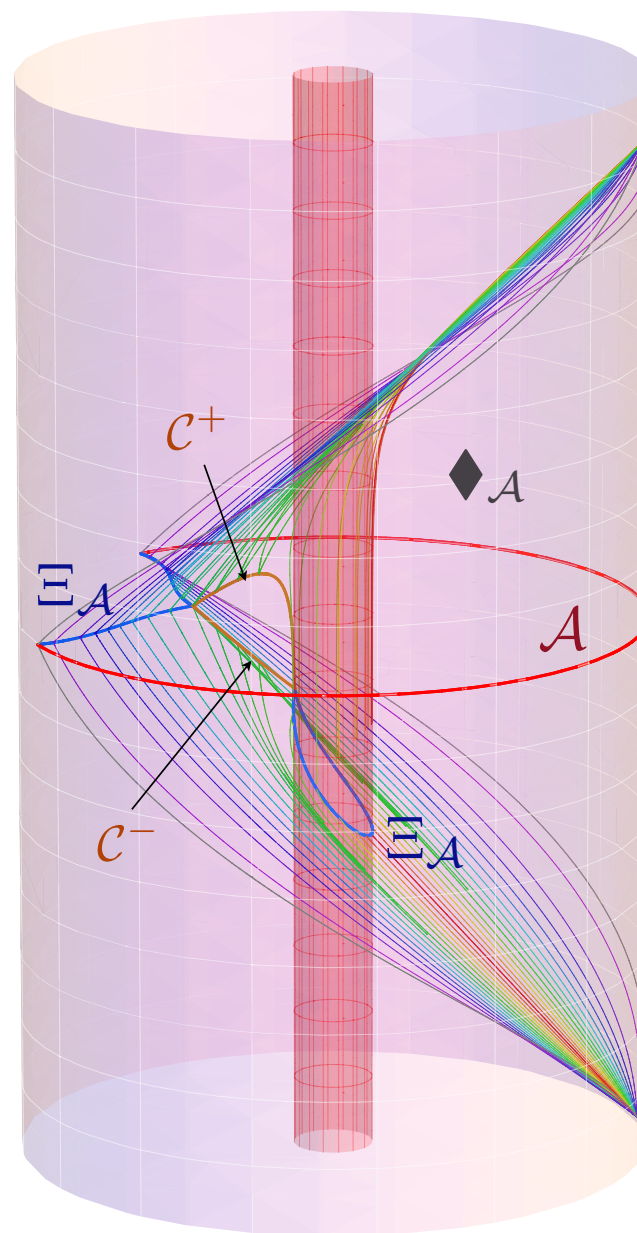
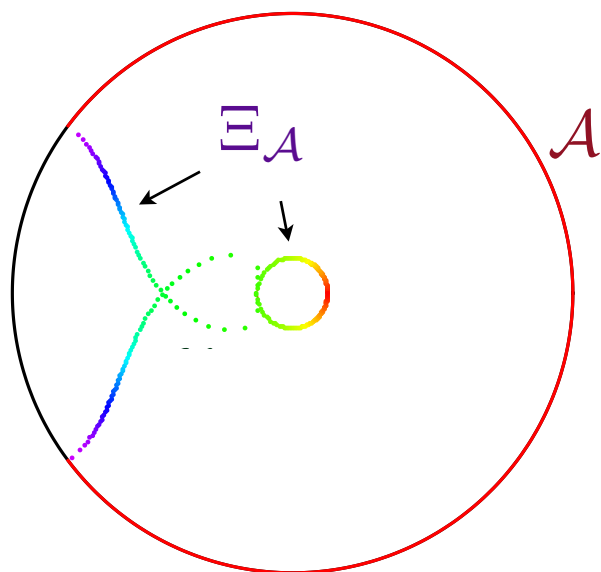
Schwarzschild-AdS₅

[Hubeny, Rangamani, E.T., JHEP (2013)]

$$r_+ = 1$$



$$r_+ = 0.2$$



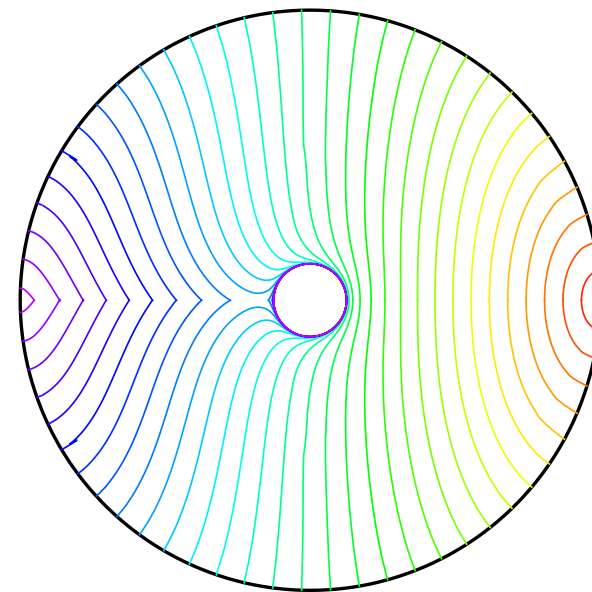
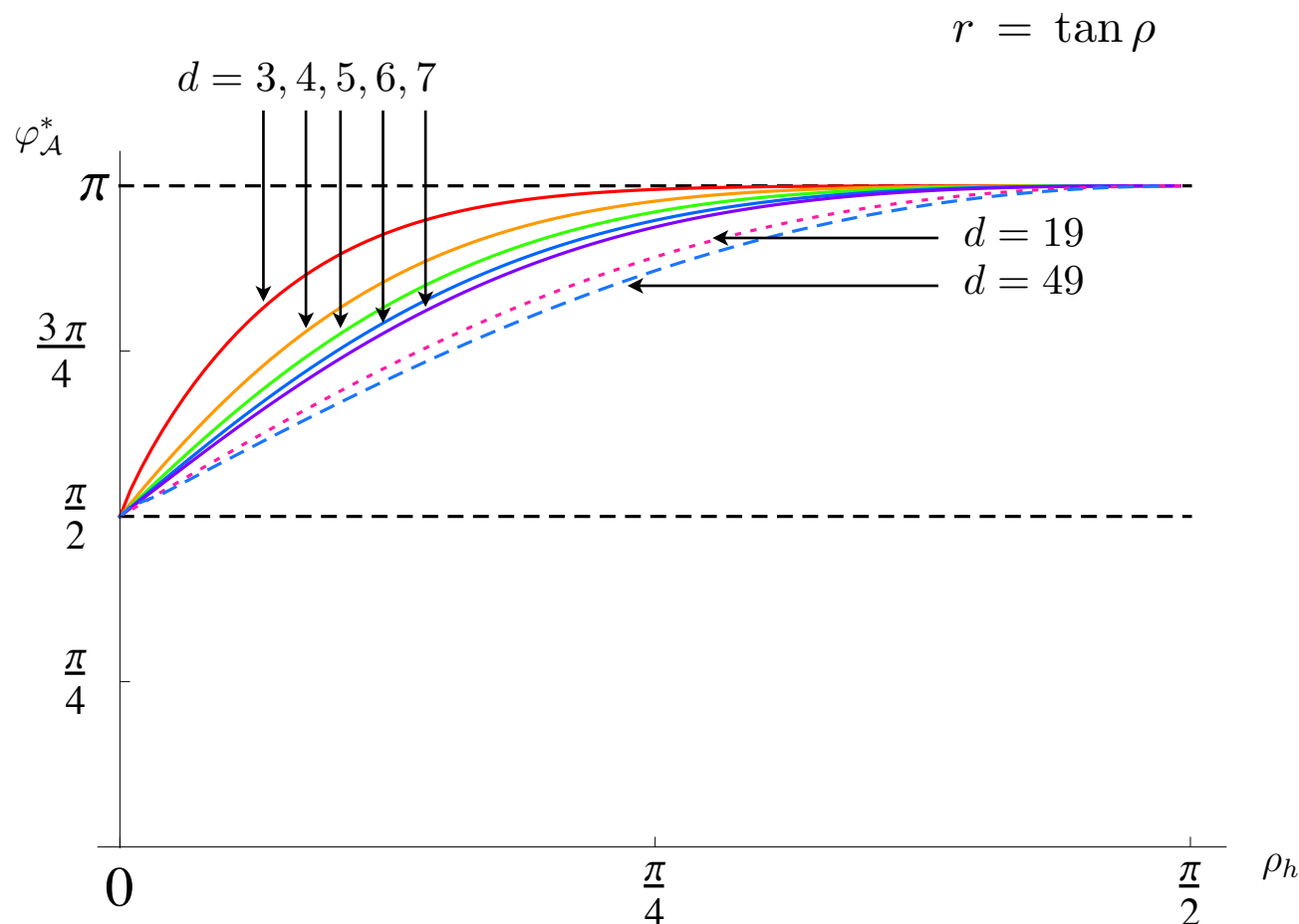
CHI: Schwarzschild-AdS black hole

[Hubeny, Rangamani, E.T., JHEP (2013)]

Schwarzschild-AdS_{d+1}

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 (d\varphi^2 + \sin^2 \varphi d\Omega_{d-2}^2)$$

$$f(r) = r^2 + 1 - \frac{r_h^{d-2} (r_h^2 + 1)}{r^{d-2}}$$



Critical angle $\varphi_{\mathcal{A}}^*$
where $\Xi_{\mathcal{A}}$ pinches off

One-point entropy

A proposal for the CFT quantity dual to the causal mutual information

[Kelly, Wall, 1309.3610]

- Coarse grained entropy $\mathcal{S}_{\mathcal{A}}$ associated to $\mathcal{A}$

[Gell-Mann, Hartle, PRA (2007)]

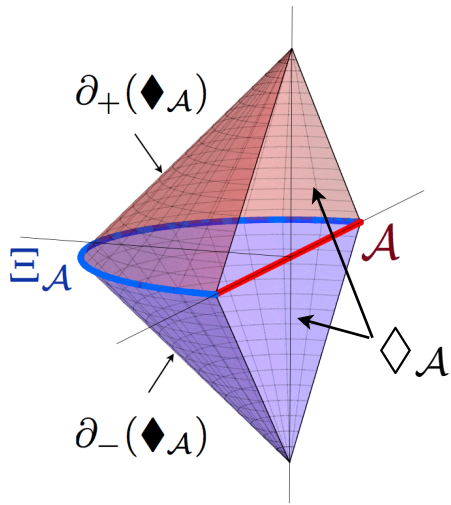
$$\mathcal{S}_{\mathcal{A}} = \sup_{\tau_{\mathcal{A}} \in T_{\mathcal{A}}} [S_{\mathcal{A}}(\tau_{\mathcal{A}})]$$

$S_{\mathcal{A}}(\tau_{\mathcal{A}})$ Von Neumann entropy of $\tau_{\mathcal{A}}$

$T_{\mathcal{A}}(\rho_{\mathcal{A}})$ set of all the density matrices $\tau_{\mathcal{A}}$ s.t.

$$\text{Tr}[O_m \tau_{\mathcal{A}}] = \text{Tr}[O_m \rho_{\mathcal{A}}]$$

$\{O_m\}$ set of operators supported in $\diamond_{\mathcal{A}}$



- The one-point entropy $\mathcal{S}_{\mathcal{A}}^{(1)}$ is defined as $\mathcal{S}_{\mathcal{A}}$ when $\{O_m\}$ are the one-point functions of all gauge-invariant, local operators supported in $\diamond_{\mathcal{A}}$
- Some properties (e.g. additivity) have been checked but quantitative checks are missing

Vaidya metrics

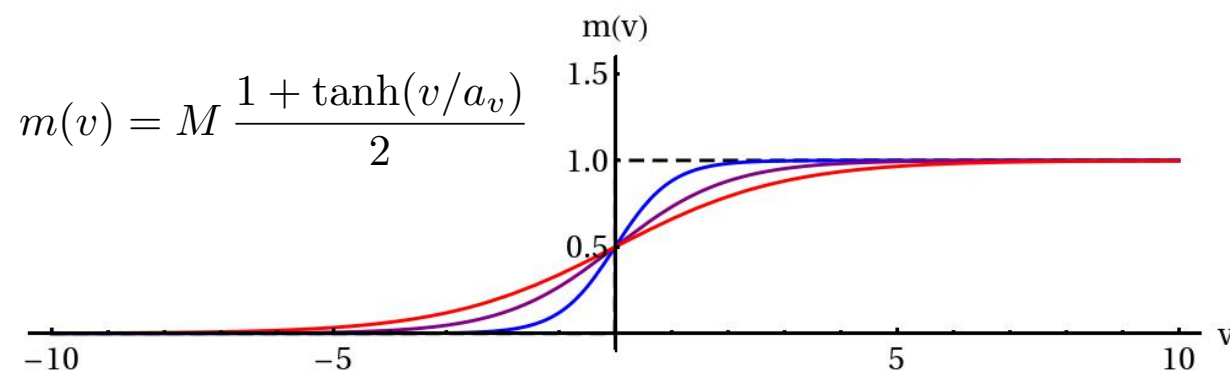
- Formation of a black hole through the collapse of null dust
 $d + 1$ dimensional Vaidya-AdS $_{d+1}$

$$ds^2 = 2 dv dr - f(r, v) dv^2 + r^2 d\Sigma_{d-1, K}^2$$

$$f(r, v) = r^2 \left(1 + \frac{K}{r^2} - \frac{m(v)}{r^d} \right)$$

$$T_{vv} \propto \frac{1}{r^{d-1}} \partial_v m(v) \quad \Rightarrow \quad \text{Null energy condition } T_{\mu\nu} N^\mu N^\nu \geq 0$$

imposes that $\partial_v m(v) \geq 0$



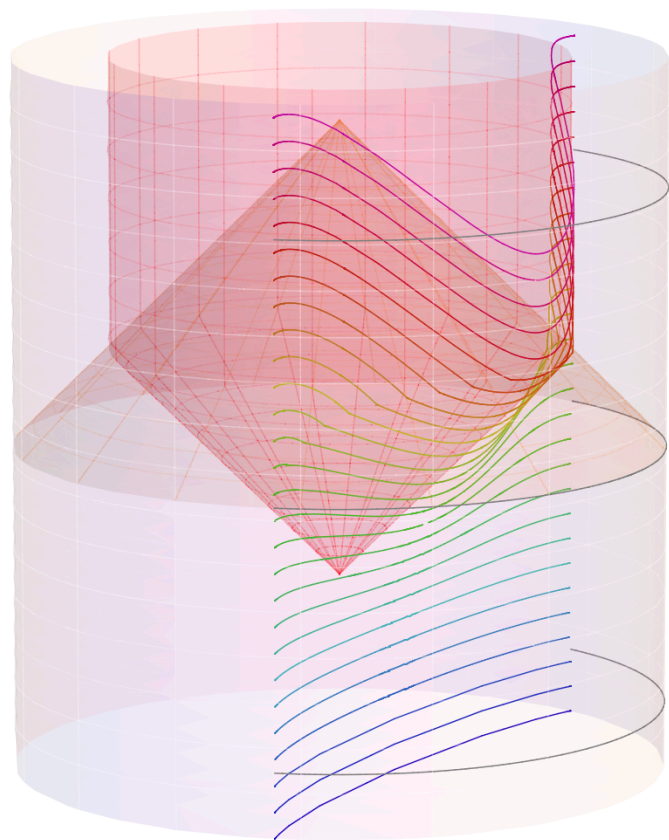
interpolates between AdS $_{d+1}$
and Schwarzschild-AdS $_{d+1}$

- Holographic entanglement entropy [Hubeny, Rangamani, Takayanagi, JHEP (2007)]
[Abajo-Arrestia, Aparicio, Lopez, JHEP (2010)]
- Thin shell limit $a_v \rightarrow 0$
(analytic solution for $d = 2$) [Balasubramanian et al., PRL, PRD (2011)]
[Liu, Suh, PRL, PRD (2013)]

CHI in the thin shell limit

[Hubeny, Rangamani, E.T., JHEP (2013)]

Vaidya-AdS₃ in the thin shell limit ($m = m_0 \theta(v)$)



IV
 $t_{\mathcal{A}} = \varphi_{\mathcal{A}}$

Thermal

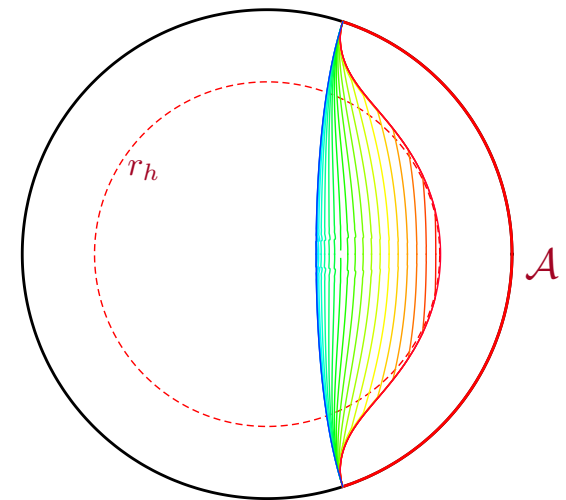
III
 $t_{\mathcal{A}} = 0$

Shell encounter by $\Xi_{\mathcal{A}}$
(both $\partial_+(\diamond_{\mathcal{A}})$ and $\partial_-(\diamond_{\mathcal{A}})$ refract through the shell)

II
 $t_{\mathcal{A}} = -\varphi_{\mathcal{A}}$

Shell encounter by $\partial_+(\diamond_{\mathcal{A}})$ only

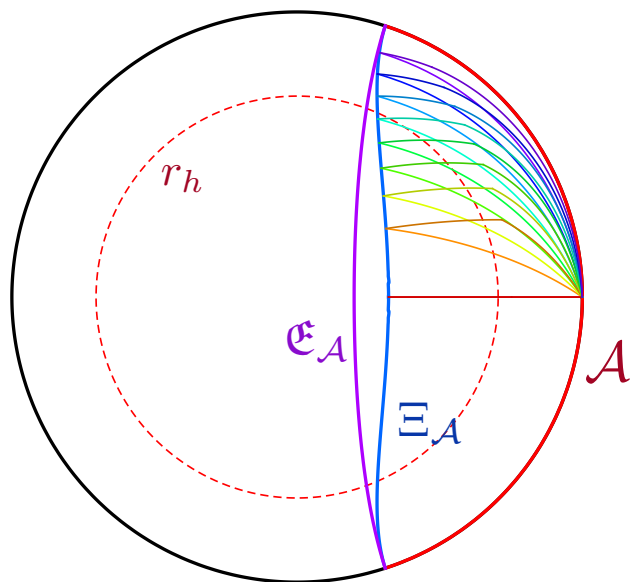
I
Vacuum



CHI in the thin shell limit

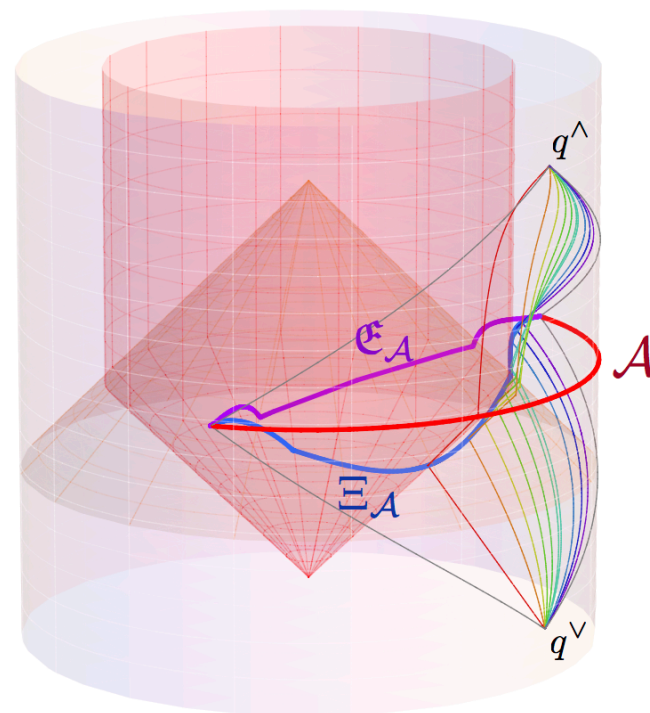
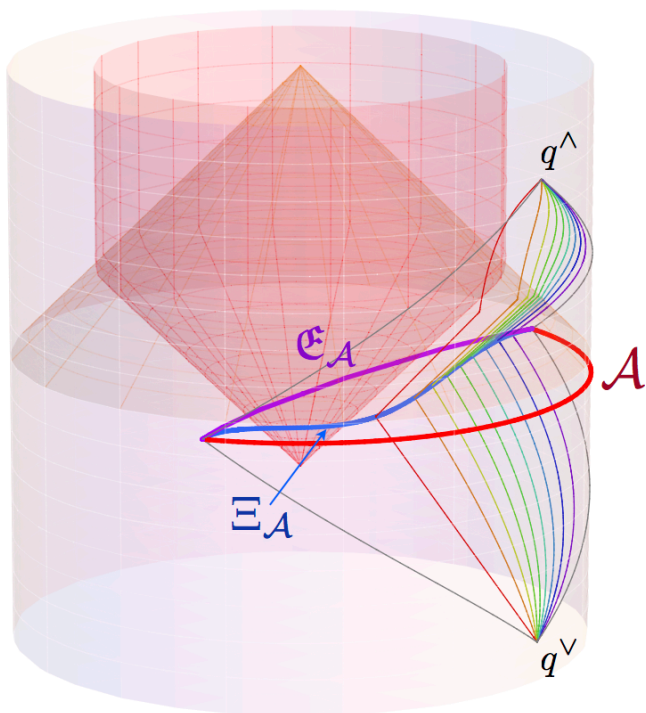
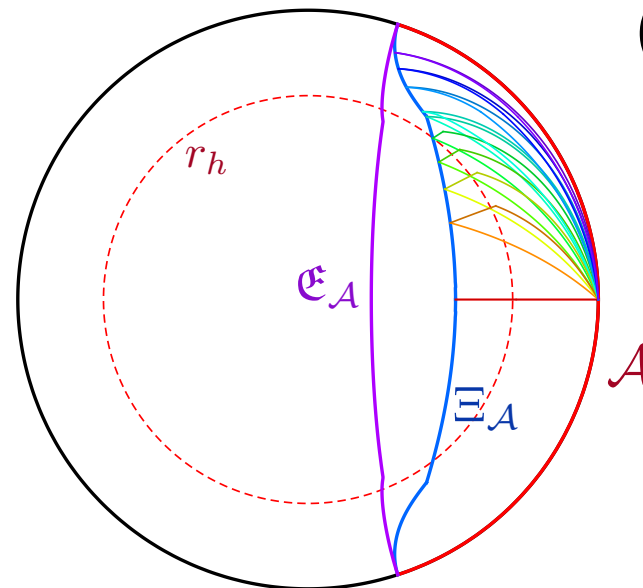
(II)

$$-\varphi_A < t_A < 0$$

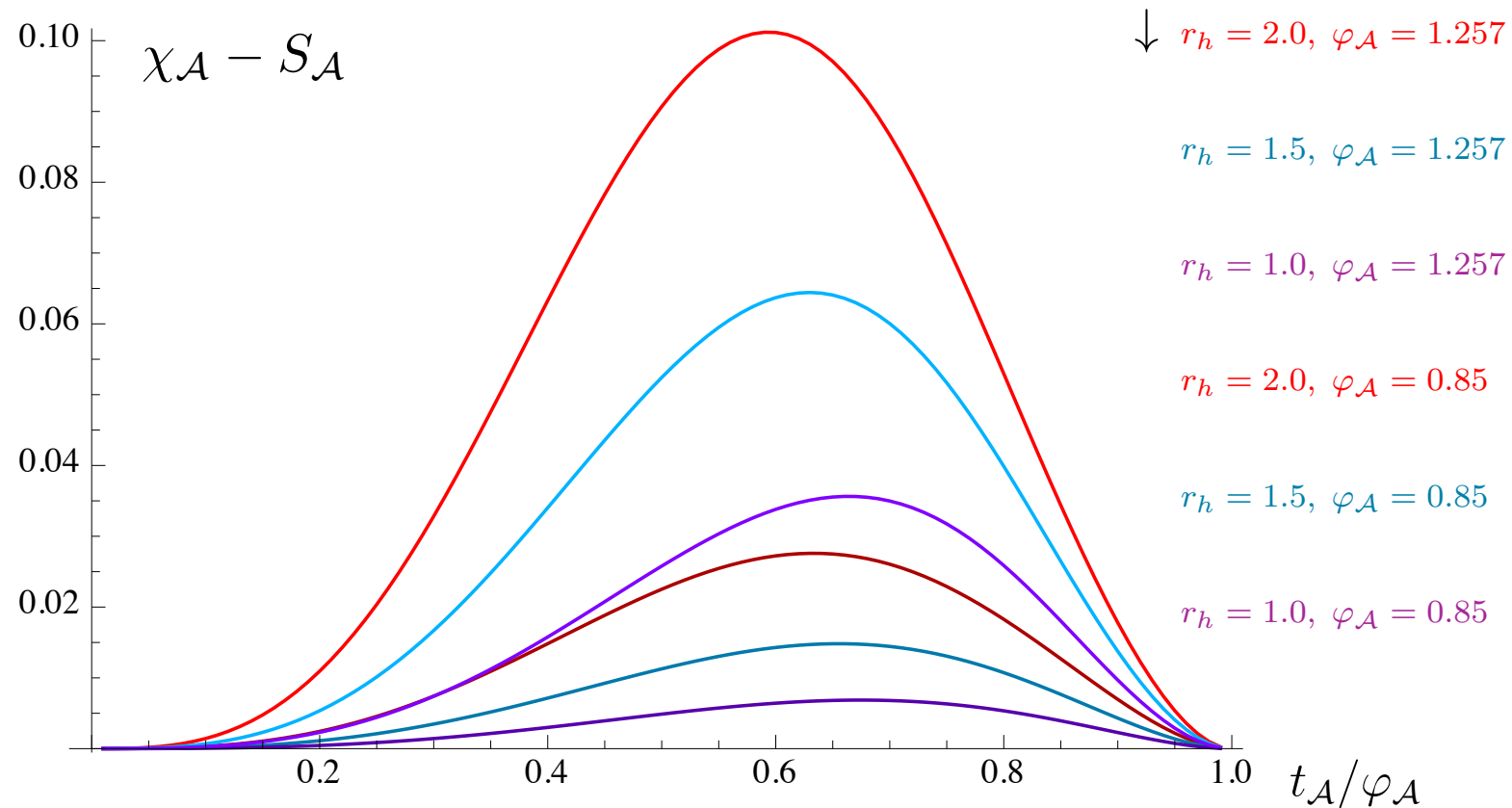


(III)

$$0 < t_A < \varphi_A$$



Thin shell limit in 3 dim: HEE & CHI



$$\chi_{\mathcal{A}} - S_{\mathcal{A}} = \frac{c_{\text{eff}}}{24} (\cot \varphi_{\mathcal{A}} - r_h \coth r_h \varphi_{\mathcal{A}}) (r_h^2 \operatorname{csch}^2 r_h \varphi_{\mathcal{A}} - \csc^2 \varphi_{\mathcal{A}} - r_h^2 - 1) t_{\mathcal{A}}^3 + \dots$$

$$\chi_{\mathcal{A}} - S_{\mathcal{A}} = \frac{c_{\text{eff}}}{6} \frac{(r_h^2 + 1)^2}{3 r_h^2 \coth^2(r_h \varphi_{\mathcal{A}})} (\varphi_{\mathcal{A}} - t_{\mathcal{A}})^2 + \dots$$

Conclusions & Open issues

- Entanglement entropies for *disjoint intervals*: $\text{Tr} \rho_A^n$ for the simplest CFTs
The $N \geq 2$ intervals case includes all the data of the CFT
- *Entanglement for mixed states.*
Negativity in QFT (1+1 CFTs): $\text{Tr}(\rho^{T_2})^n$ and $\mathcal{E}$
- Holography can suggest new measures of entanglement (e.g. CHI)

 Analytic continuations, Negativity for fermions,
Higher dimensions, Interactions, Holographic interpretations, ...

- Entanglement & RG [Casini, Huerta, (2004), (2012)] [Myers, Singh, (2012)] [...]
- Quantum Hall Effect [Li, Haldane, (2008)] [...]
- Excited states [Berganza, Alcaraz, Sierra, (2011)] [...]

Thank you!