Gravitation from entanglement in holographic CFTs

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Motivation

$$\boxed{\text{What is (quantum) gravity?}}$$

$$\Big\downarrow \text{ key}$$

entropy → geometric interpretation

Geometric entropy

- black hole entropy

$$S_{BH} = \frac{A_{horizon}}{4G} \quad \Rightarrow \quad \text{quantum gravity is holographic}$$

- entanglement entropy in AdS/CFT

geometric interpretation $\longleftrightarrow$ linearized Einstein eqn. (AdS)

$\nwarrow\!\!\searrow$ $\qquad\qquad$ $\nearrow\!\!\swarrow$

entanglement "1st law"

Black hole thermodynamics

- laws of black hole mechanics $\leftrightarrow$ thermodynamics

$\swarrow$ e.o.m.

1st: $\frac{\kappa}{8\pi}\,\delta A_H = \delta M$

2nd: $\delta A_H \geq 0$

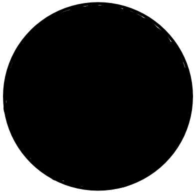
$\Big\}$ $\quad A_H \leftrightarrow S$, $\quad \kappa \leftrightarrow T$

- Hawking: $T_H = \frac{\kappa}{2\pi}$ $\Rightarrow$ $S_{BH} = \frac{A}{4}$ statistical interpretation!

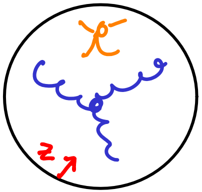
- Jacobson : $T\delta S = \delta Q$ for all local Rindler horizons $\Rightarrow$ Einstein equations $\rightarrow$ eqns of state

- Wald: higher derivatives $\rightarrow$ $S_{BH} = \int_{\text{horizon}} \frac{\partial \mathcal{L}}{\partial R_{abcd}}\,\epsilon_{ab}\,\epsilon_{cd}$

Holography



$$S_{BH} = \frac{A_H}{4G} > S \rightarrow \text{holography}$$



- all gravitational processes in bulk $\leftrightarrow$ field theory on the boundary
- concrete realization : AdS_{d+1}/CFT_d $\rightarrow$ universal !
  - CFT $\rightarrow$ fundamental
  - z direction, gravity $\rightarrow$ emergent $\rightarrow$ How?

    (large N, strong coupling)

Plan

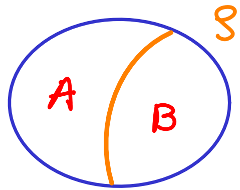
- background
- our statement
- hyperbolic "black holes" and the first law
- first law of entanglement ⇒ e.o.m

  → holographic dictionary
- examples

Background

Entanglement entropy

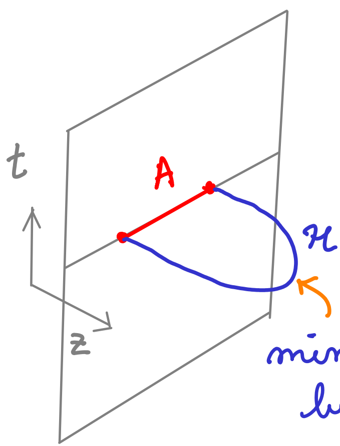
- definition :



$$S_A = -\mathrm{Tr}_A\, \rho_A \ln \rho_A$$

$$\rho_A = \mathrm{Tr}_B\, \rho$$

- in holography → geometric interpretation



minimal bulk surface

$$\boxed{S_A^{grav} = \frac{A_{min}}{4G}}$$

- Einstein gravity

(Ryu - Takayanagi)

- higher derivatives : Wald + + extrinsic curvature corrections

(Dong, Camps)

The "1st law" of entanglement

- modular Hamiltonian:

$$\rho_A \equiv \frac{e^{-H_A}}{\mathrm{Tr}\, e^{-H_A}} \qquad \leftarrow \text{usually non-local}$$

- entanglement "first law": $\rho_A \to \rho_A + \delta\rho_A$

$$\delta S_A = \mathrm{Tr}_A\left(\delta\rho_A H_A\right) = \delta\langle H_A\rangle$$

A simple modular Hamiltonian

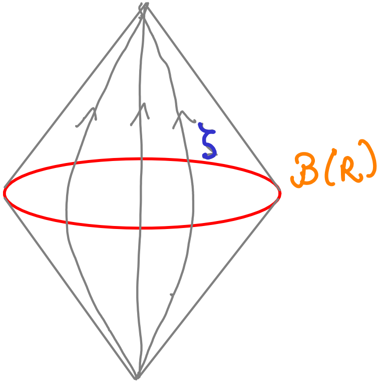
- quantum system : CFT_d ; $\rho_{total} = |0\rangle\langle 0|$

- A = ball of radius R, center $\vec{x}_0$: $B(R,\vec{x}_0)$, time t_0

- then :

$$\mathcal{H}_B = 2\pi \int_{B(R,x_0)} d^{d-1}x \, \frac{R^2 - |\vec{x}-\vec{x}_0|^2}{2R} \, T_{tt}(t_0,\vec{x})$$

$$= 2\pi \int_{B(R,x_0)} d\Sigma^\mu \, T_{\mu\nu} \, \xi^\nu$$



$$\xi = \frac{1}{2R}\left[(R^2 - t^2 - |\vec{x}-\vec{x}_0|^2)\partial_t - 2t(x^i - x_0^i)\partial_i \right]$$

Our statement

Assumptions

- CFT_d w/ holographic description (large N, strong coupling)
- $|0\rangle \to$ Poincaré AdS_{d+1} $\quad ds^2 = \frac{\ell^2}{z^2}\left(-dt^2 + dx_i dx^i + dz^2\right)$
- $\delta g \to$ small metric perturbation $h_{ab}(x^\mu, z)$
- $\delta S_B \to$ Wald functional of h_{ab} $\leftarrow$ no assumptions!

We show:

universal!

$$\boxed{\delta S_B = \delta\langle H_B\rangle} \quad \Longleftrightarrow \quad \boxed{\text{linearized Einstein equations}}$$

$\forall R, \vec{x}^0$, Lorentz frame

+ holographic dictionary

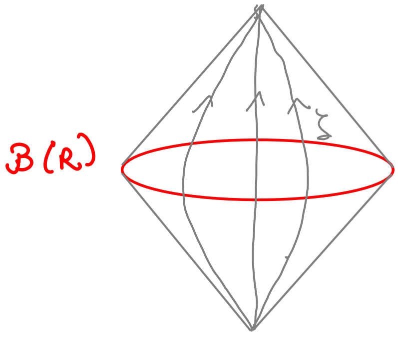
Hyperbolic "black holes"

and $\delta S_B = \delta \langle \mathcal{H}_B \rangle$

The ball modular Hamiltonian

$$\mathcal{H}_B = 2\pi \int_{B(R,x^0)} d\Sigma^\mu \, T_{\mu\nu}(x)\, \zeta^\nu$$

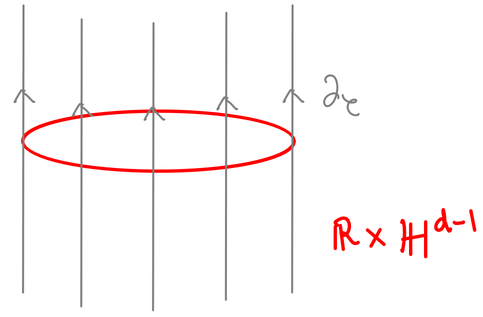
CFT, $|0\rangle$



conformal

$\longleftrightarrow$

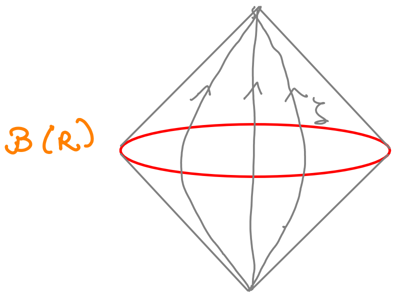
map



$$\zeta = \frac{1}{2R}\left[(R^2 - t^2 - |\vec{x}|^2)\partial_t - 2tx^i\partial_i \right] \longleftrightarrow \partial_\tau \qquad (\mathcal{H}_\zeta = \mathcal{H}_\tau)$$

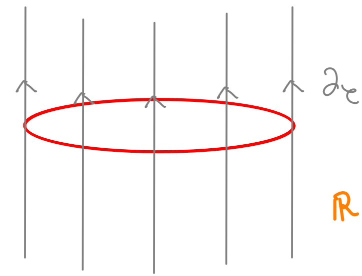
$$\rho_B \longleftrightarrow \rho_{thermal} \qquad (S_B = S_T)$$

CFT_d



$\xi \leftrightarrow \partial_\tau$

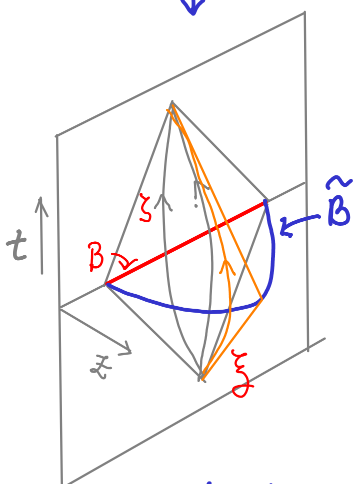
conformal $\longleftrightarrow$ map



$\mathbb{R} \times \mathbb{H}^{d-1}$

AdS/CFT $\downarrow$

AdS/CFT $\downarrow$



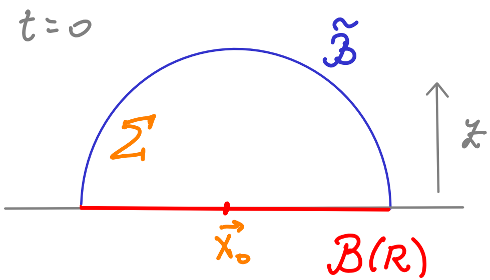
coordinate $\longleftrightarrow$ transf.

Hyperbolic "black hole"

$$ds^2 = -\frac{\rho^2 - \ell^2}{R^2} d\tau^2 + \frac{\ell^2 d\rho^2}{\rho^2 - \ell^2} + \rho^2 ds^2_{\mathbb{H}^{d-1}}$$

AdS Killing vect $\xi \leftrightarrow \partial_\tau$

First law for hyperbolic "black hole"



- $\tilde{\mathcal{B}}$: horizon (Killing, ξ)

$$z^2 + |\vec{x} - \vec{x}_0|^2 = R^2$$

- Σ : b.h. exterior

$$\boxed{S^{\text{Wald}}_{\text{Hyp. B.H.}} = \delta E[\partial_t]} \iff \boxed{\delta S^{\text{grav}}_{\mathcal{B}} = \delta E^{\text{grav}}_{\mathcal{B}}[\xi]}$$

Wald ← thermo

entang ↳ $\delta S^{\text{grav}}_{\mathcal{B}}$

first law of black hole thermodynamics (Iyer-Wald) ↔ first law of entanglement

Summary

- map to hyperbolic black hole:

  1st law entanglement = 1st law b.h. thermodyn.

  Iyer - Wald

- check of holographic consistency:

  - e.o.m $\Rightarrow$ $\delta S_B = \delta\langle H_B\rangle$ in the CFT

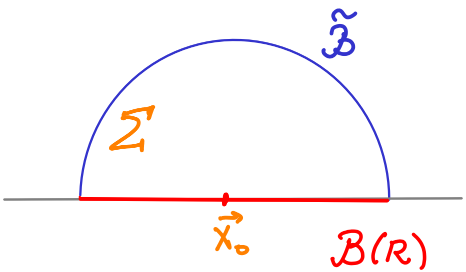
  - check holographic interpretation of δS_B, δE_B is correct

- previous work: Blanco, Casini, Hung, Myers

$$\delta S_B = \delta \langle \mathcal{H}_B \rangle \quad \Rightarrow \quad \text{e.o.m}$$

The Wald-Iyer proof

- space-time background (ϕ) w/ Killing vect ξ (K.H.)
- $\delta\phi$ on-shell $\Rightarrow$ $d\big(\delta \overset{(n-2)}{Q_\xi} - \xi\cdot\Theta(\delta\phi)\big) = 0$
- $Q_\xi = -\dfrac{\partial \mathcal{L}}{\partial R_{abcd}}\, \varepsilon_{ab}\, \nabla_{[c}\xi_{d]} + \xi^c W_c$



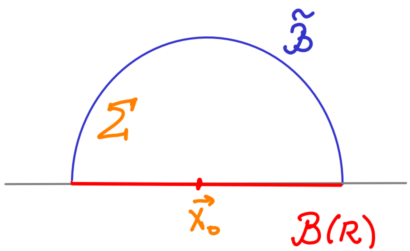
$$0 = \int_\Sigma d\big(\delta Q_\xi - \xi\cdot\Theta(\delta\phi)\big) =$$

$$= \underbrace{\int_{\tilde{B}} \delta Q_\xi}_{\delta S_{\text{Wald}}} - \underbrace{\int_B \delta Q_\xi - \xi\cdot\Theta(\delta\phi)}_{\delta E[\xi]}$$

Converse of Iyer-Wald

- $\mathcal{L}_\xi \phi = 0$, $\delta\phi$ not on-shell
- $\chi \equiv \delta Q_\xi - \xi \cdot \Theta(\delta\phi) \rightarrow d\chi = -\xi^a \delta E^g_{ab} \, \varepsilon^b$

$(\varepsilon^b = \varepsilon^b{}_{a_1 \dots a_{n-1}} dx^1 \wedge \dots \wedge dx^{n-1})$



$$\int_\Sigma d\chi = \int_{\tilde{B}} \delta Q_\xi - \int_B \delta Q_\xi - \xi \cdot \Theta(\delta\phi)$$

$$\int_\Sigma \xi^t \delta E^g_{tt} = \delta S^{grav}_B - \delta E^{grav}_B = 0$$

$$\int_{\Sigma(R, \vec{x}_0)} \xi^t \delta E^g_{tt} \, d^d x = 0 \quad \forall R, \vec{x}_0 \Rightarrow \boxed{\delta E^g_{tt} = 0}$$

Technical comments

- request $\delta S_B = \delta E_B$ for all boosted balls $\Rightarrow$ $\delta E^{g}_{\mu\nu} = 0$

  bnd directions

- $z\mu$ & zz components $\rightarrow$ Noether identities

  $+$ $z=0$ constraints *

- can show $E_\xi^{\text{Wald}} = E_\xi^{\text{boundary}\,*}$ (holographic stress tensor)

- need specific falloff* of perturbation for a well-defined E_ξ^{Wald}

More comments

- works for all gravitational theories → universal
- if we can compute S_B^{CFT} and show $S_B[h]$

  ⇒ Einstein's eqns are the 1st law relation.

- previous work: Lashkari, McDermott, van Raamsdonk

  area law → Einstein eqn. 4d

The holographic dictionary from entanglement

- modular Hamiltonian for $B(R, \vec{x}_0)$

$$\mathcal{H}_B = 2\pi \int_{B(R,x_0)} d^{d-1}x \; \frac{R^2 - |\vec{x}-\vec{x}_0|^2}{2R} \; T_{tt}(t_0, \vec{x})$$

- take $R \to 0$ limit: $T_{tt}(t_0, \vec{x}) \to T_{tt}(t_0, \vec{x}_0)$

$$\delta\langle \mathcal{H}_B \rangle = 2\pi \, \delta\langle T_{tt}(x_0) \rangle \int_{|\vec{x}|<R} d^{d-1}x \; \frac{R^2 - |\vec{x}|^2}{2R} = \frac{2\pi R^d \Omega_{d-2}}{d^2-1} \, \delta\langle T_{tt}(x_0)\rangle$$

- $\delta\langle H_B \rangle = \delta S_B[h]$ => holographic dictionary

$$\boxed{\delta\langle T_{tt}(x_0)\rangle = \frac{d^2-1}{2\pi \Omega_{d-2}} \lim_{R\to 0} \left[R^{-d} S_{B(R,x^0)}[h] \right]}$$

Examples

Area law

- Poincaré AdS + $\quad ds^2 = \frac{\ell^2}{z^2}\left(-dt^2 + dx_i dx^i + dz^2\right)$

  perturbation $h_{\mu\nu}(z, x^\lambda)$ $\quad h_{zz} = h_{z\mu} = 0$

- assumption: $S_B = \frac{\text{Area}(\tilde{B})}{4 G_{d+1}}$

$$\delta S_B = \frac{R}{8 G_{d+1}} \int_{B(R,x_0)} d^{d-1}x \; z^{2-d} \left(\delta^{ij} - \frac{\Delta x^i \Delta x^j}{R^2}\right) h_{ij}$$

Δx^i: $x^i - x_0^i$

- as $R \to 0$, need $\delta S_B \propto R^d \Rightarrow$

$$h_{ij}(z, x^\lambda) \propto z^{d-2} h_{ij}^{(d)}(x^\lambda) + \dots$$

falloff!

The holographic stress tensor

- $$\lim_{R\to 0}\left(R^{-d} S_{B(R)}[h]\right) = \frac{d\,\Omega_{d-2}}{8(d^2-1)\,G_{d+1}}\; h^{(d)\,i}{}_{i}(x^0)$$

- $$\delta\langle T_{tt}\rangle = \frac{d}{16\pi G_{d+1}}\, h^{(d)\,i}{}_{i} = \frac{d}{16\pi G_{d+1}}\left[-\eta_{00}\, h^{(d)\,\mu}{}_{\mu} + h_{00}\right]$$

- all Lorentz frames :

holographic stress T

Einstein!

$$\delta\langle T_{\mu\nu}\rangle = \frac{d}{16\pi G_{d+1}}\left(h^{(d)}_{\mu\nu} - \eta_{\mu\nu}\, h^{(d)\,\lambda}{}_{\lambda}\right)$$

- conservation :

$$\left.\begin{array}{l} T^{\mu}{}_{\mu} = 0 \;\Rightarrow\; h^{(d)\,\mu}{}_{\mu} = 0 \\ \partial_\mu T^{\mu}{}_{\nu} = 0 \;\Rightarrow\; \partial_\mu h^{(d)\,\mu}{}_{\nu} = 0 \end{array}\right\}$$

$z=0$ constraints

The FG expansion

- $\delta\langle H_B\rangle = 2\pi \int_{B(R,x^0)} d^{d-1}x \, \frac{R^2 - |\vec{x}-\vec{x}_0|^2}{2R} \, \delta\langle T_{tt}(t_0,\vec{x})\rangle$

- $T_{tt}(\vec{x}) = T_{tt}(\vec{x}_0) + \underbrace{\Delta x^i}_{\sim R} \partial_i T_{tt}(\vec{x}_0) + \frac{1}{2} \underbrace{\Delta x^i \Delta x^j}_{\sim R^2} \partial_i \partial_j T_{tt}(\vec{x}_0) + \ldots$

- after integration : $\delta\langle H_B\rangle = R^d \left(\# \, \delta T_{tt} + \# \, R^2 \vec{\Box} \, \delta T_{tt} + \ldots \right.$

- $\delta S_B[h] = \delta\langle H_B\rangle \implies$

$$\left.\begin{array}{l} \bullet\ h_{\mu\nu} = z^{d-2} \sum_{n=0}^{\infty} z^{2n} h_{\mu\nu}^{(d+2n)} \\ \bullet\ h_{\mu\nu}^{(d+2n+2)} \propto \Box \, h_{\mu\nu}^{(d+2n)} \end{array}\right\} \leftarrow \text{linearized FG expansion}$$

using constraints.

Polynomial Wald functionals

$\delta S_B = \delta S_B^{Wald}$

- $S_B^{Wald} = \int_{\tilde{B}} \frac{\partial \mathcal{L}}{\partial R_{abcd}} \varepsilon^{ab} \nabla_c \xi_d \, d\Sigma$

useful param.

- $\mathcal{L} = \frac{1}{16\pi G_{d+1}} \left(\sigma R - 2\Lambda + a_1 R_{abcd} R^{abcd} + a_2 R_{ab} R^{ab} + a_3 R^2 \right)$

- since linearized → models all polynomial Wald funct

- $\delta S_B \propto R^d$ as $R \to 0$ => $h_{\mu\nu}(z, x^\lambda) \propto z^{d-2} h^{(d)}_{\mu\nu}(x^\lambda) + \ldots$

$$\langle T_{\mu\nu} \rangle = \frac{d\, \ell^{d-5}}{16\pi G_{d+1}} \left[\sigma \ell^2 + 4a_1(d-3) - 2d\, a_2 - 2d(d+1) a_3 \right] h^{(d)}_{\mu\nu}$$

- agrees w/ Hohm/Tonni Smolic/Taylor

$\partial_z \sim \mathcal{O}(R^{-1})$, $\partial_{x^\lambda} \sim \mathcal{O}(1)$

Comments

- easy to calculate! (no variational principle)
- universal result dictated by scaling $\Rightarrow$

  $\Rightarrow$ it must also hold at non-linear level

  - relevant expansion parameter is $\langle T_{\mu\nu} \rangle R^d$
  - can be used to compute $\langle T_{\mu\nu} \rangle$ in R^3 theories

    and higher

An R^4 example

- $$S = \frac{1}{16\pi G_{d+1}} \int d^{d+1}x \sqrt{g} \left[R - 2\Lambda + \alpha (R_{abcd} R^{abcd})^2 \right]$$

Smolic - Taylor ($d=3$)

- black holes satisfy first law thermodynamics if $\partial_\alpha M = 0$

- effective R^2 theory w/: $a_1 = \dfrac{4\alpha d(d+1)}{\ell^4}$, $a_2 = 0$, $a_3 = \dfrac{16\alpha}{\ell^4}$

$$\sigma = 1 + \frac{32\alpha d(d+1)}{\ell^6}$$

- $$\langle T_{00} \rangle = \frac{d\,\ell^{d-3}}{16\pi G_{d+1}} \left(1 + \frac{16\alpha d(d+1)(d-3)}{\ell^6} \right) h_{00}^{(d)}$$

- $\partial_\alpha M = 0$ for $d = 3$!

Further comments

- $\langle T_{\mu\nu} \rangle \propto h_{\mu\nu}^{(d)} \quad \Rightarrow \quad h^{\mu}{}_{\mu}{}^{(d)} = 0 \;, \; \partial_\mu h^{\mu}{}_{\nu}{}^{(d)} = 0$

  in any higher derivative theory

- Fefferman - Graham expansion

- new solutions to $\delta S_B = \delta \langle H_B \rangle$ satisfying $\delta S = 0$

  $h_{\mu\nu} \sim z^{\Delta - 2} h_{\mu\nu}^{(\Delta)} + \dots \qquad \Delta \to \infty$ as $a_i \to 0$

- Δ satisfies cubic eqn. when $a_i \neq 0$, $\Delta = 0$ always a soln.

- entanglement first law $\to$ all solutions of the gravity theory

Summary & outlook

- if $S_B = S_B[\text{bulk metric}]$, then $\delta S_B = \delta\langle H_B\rangle \Leftrightarrow$ lin. Einstein eqns.
- $\delta S_B = \delta\langle H_B\rangle \to$ new method for deriving holographic stress tensor
- non-linear generalization? $\to$ Jacobson
- other holographic backgrounds?
- quantum states in the bulk?

Thank you!